Tetrahedron equations associated with quantized six-vertex models

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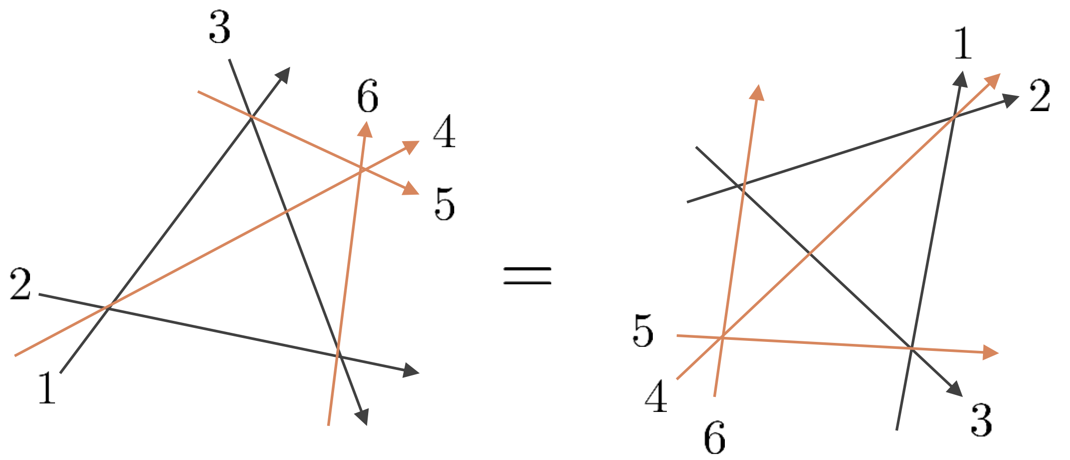
Based on: A. Kuniba, S. Matsuike, AY, arXiv:2208.10258

Outline

Tetrahedron equation



- Matrix equation on $V_1 \otimes \cdots \otimes V_6$ (V_i: linear space) [Zamolodchikov'81]
  - X_{ijk} $(X = A, B, C, D)$ acts non-trivially only on $V_i \otimes V_j \otimes V_k$.

$$A_{124}B_{135}C_{236}D_{456} = D_{456}C_{236}B_{135}A_{124}$$

- 3D analog of Yang-Baxter equation (YBE)
  - We can construct a 3D version of transfer matrices similarly to YBE.
- Several solutions are known although less systematic than YBE.

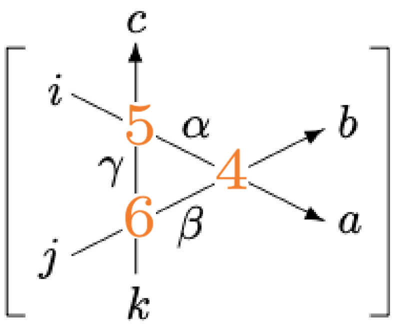
  Zamolodchikov, Baxter, Bazhanov, Korepanov, Mangazeev, Sergeev, Stroganov, Kapranov, Voevodsky, Kazhdan, Soibelman, Carter, Saito, Kuniba, Okado, ...

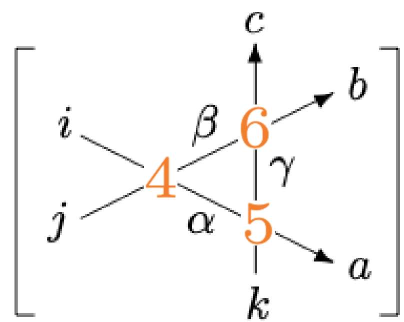
$RLLL$ relation

- Today, we focus on the $RLLL$ type tetrahedron equation:

$$L_{124}L_{135}L_{236}R_{456} = R_{456}L_{236}L_{135}L_{124}$$

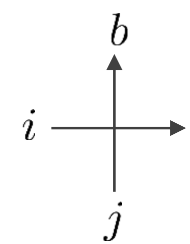
- If we specify the outer lines for 1,2,3-th spaces, this reads as

$\sum_{\alpha,\beta,\gamma}$


$\circ R_{456} = \sum_{\alpha,\beta,\gamma} R_{456} \circ$


$\cdots(*)$

$$L(|i\rangle \otimes |j\rangle \otimes |k\rangle) = \sum_{a,b} |a\rangle \otimes |b\rangle \otimes L_{i,j}^{a,b}\,|k\rangle$$



$= L_{i,j}^{a,b}$

- For each (i, j, k, a, b, c), $(*)$ gives linear equations for R.
- If we can ansatz ``good" Ls, we can then obtain a solution to the $RLLL$ type tetrahedron equation by solving these equations.
- In fact, it can be done by considering a quantized six vertex model for Ls.

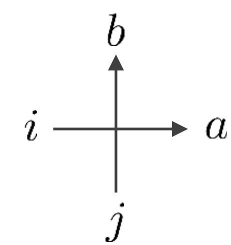
q-Oscillator algebra valued six vertex model

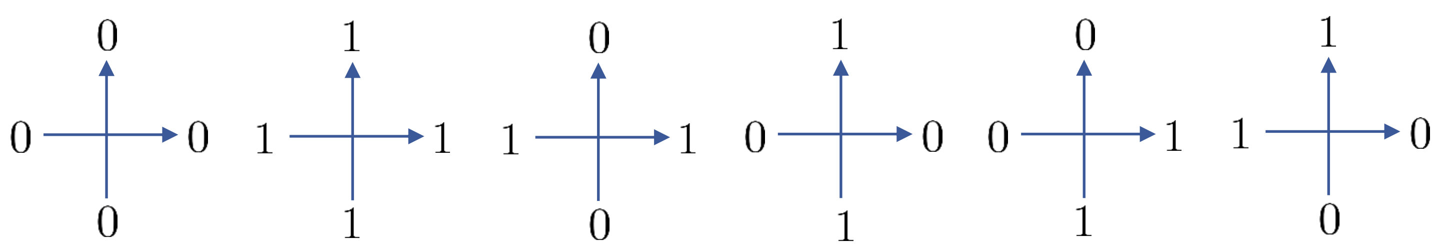
- q-Oscillator algebra O_q
  - Genetators: $\mathbf{k}, \mathbf{a}^{\pm}$
  - Relations $\mathbf{k}\mathbf{a}^{\pm} = q^{\pm}\mathbf{a}^{\pm}\mathbf{k}, \quad \mathbf{a}^{+}\mathbf{a}^{-} = 1 - \mathbf{k}^2, \quad \mathbf{a}^{-}\mathbf{a}^{+} = 1 - q^2\mathbf{k}^2$
  - Representation π_O on $F_+ = \oplus_{m \in \mathbb{Z}_{\geq 0}} \mathbb{C}\,|m\rangle$:

$$\pi_O : \mathbf{k}\,|m\rangle = q^m\,|m\rangle\,, \quad \mathbf{a}^{+}\,|m\rangle = |m+1\rangle\,, \quad \mathbf{a}^{-}\,|m\rangle = (1-q^2)\,|m-1\rangle$$

- L-operator $L^O \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes F_+)$ [Bazhanov-Sergeev'06]

$$L^O(|i\rangle \otimes |j\rangle \otimes |k\rangle) = \sum_{a,b \in \{0,1\}} |a\rangle \otimes |b\rangle \otimes \pi_O((L^O)^{a,b}_{i,j})\,|k\rangle \quad (L^O)^{a,b}_{i,j} =$$





1	1	$\mu\mathbf{k}$	$-q\mu^{-1}\mathbf{k}$	$\mathbf{a}^{+}$	$\mathbf{a}^{-}$

μ: parameter

$RLLL$ relation for OOO

- Thm: [Bazhanov-Sergeev'06]
  - Consider the following $RLLL$ relation for L^O:

$$\underbrace{L^O_{124}}_{\mu_4}\underbrace{L^O_{135}}_{\mu_5}\underbrace{L^O_{236}}_{\mu_6}R^{OOO}_{456} = R^{OOO}_{456}L^O_{236}L^O_{135}L^O_{124}$$

  - $R^{OOO} \in \mathrm{End}(F_+^{\otimes 3})$ is uniquely determined and given by

$$(R^{OOO})^{a,b,c}_{i,j,k} = \delta^{a+b}_{i+j}\delta^{b+c}_{j+k}\left(\frac{\mu_3}{\mu_2}\right)^i\left(-\frac{\mu_1}{\mu_3}\right)^b\left(\frac{\mu_2}{\mu_1}\right)^k q^{ik+b(k-i+1)}\binom{a+b}{a}_{q^2}\,{}_2\phi_1\left(\begin{matrix}q^{-2b},q^{-2i}\\ q^{-2a-2b}\end{matrix};q^2,q^{-2c}\right)$$

$$(z;q)_\infty = \prod_{n\geq 0}(1-zq^n) \qquad (z;q)_m = \frac{(z;q)_\infty}{(zq^m;q)_\infty} \qquad {}_2\phi_1\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix};q,z\right) = \sum_{n\geq 0}\frac{(\alpha;q)_n(\beta;q)_n}{(\gamma;q)_n(q;q)_n}z^n$$

  - R^{OOO} also satisfies the $RRRR$ type tetrahedron equation:

$$R^{OOO}_{124}R^{OOO}_{135}R^{OOO}_{236}R^{OOO}_{456} = R^{OOO}_{456}R^{OOO}_{236}R^{OOO}_{135}R^{OOO}_{124}$$

- Thm: [Kapranov-Voevodsky'94]
  - R^{OOO} = intertwiner of irreps of quantum coordinate ring $A_q(A_2)$

$$R^{OOO}\circ(\pi_1\otimes\pi_2\otimes\pi_1(\Delta^{\mathrm{op}}(g))) = (\pi_2\otimes\pi_1\otimes\pi_2(\Delta(g)))\circ R^{OOO} \quad {}^\forall g\in A_q(A_2)$$

Outline

q-Weyl algebra

- Aim: Generalize the $RLLL$ approach by Bazhanov-Sergeev
- Recall: q-Oscillator algebra O_q
  - Genetators: $\mathbf{k}, \mathbf{a}^{\pm}$
  - Relations $\mathbf{k}\mathbf{a}^{\pm} = q^{\pm}\mathbf{a}^{\pm}\mathbf{k}, \quad \mathbf{a}^{+}\mathbf{a}^{-} = 1 - \mathbf{k}^2, \quad \mathbf{a}^{-}\mathbf{a}^{+} = 1 - q^2\mathbf{k}^2$
  - Representation π_O on $F_+ = \oplus_{m \in \mathbb{Z}_{\geq 0}} \mathbb{C}\,|m\rangle$:

$$\pi_O : \mathbf{k}\,|m\rangle = q^m\,|m\rangle\,, \quad \mathbf{a}^{+}\,|m\rangle = |m+1\rangle\,, \quad \mathbf{a}^{-}\,|m\rangle = (1-q^2)\,|m-1\rangle$$

- q-Weyl algebra W_q
  - Generators: $X^{\pm 1}, Z^{\pm 1}$
  - Relations: $XZ = qZX$
  - Representations π_X, π_Z on $F = \oplus_{m \in \mathbb{Z}} \mathbb{C}\,|m\rangle$:

$$\pi_X : X\,|m\rangle = q^m\,|m\rangle\,, \quad Z\,|m\rangle = |m+1\rangle \quad \text{(coordinate rep)}$$

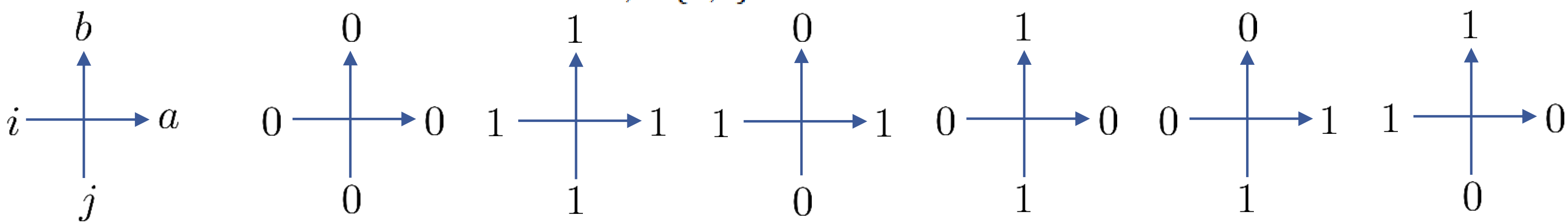
$$\pi_Z : X\,|m\rangle = |m-1\rangle\,, \quad Z\,|m\rangle = q^m\,|m\rangle \quad \text{(momentum rep)}$$

- An embedding $O_q \hookrightarrow W_q$: $\mathbf{k} \mapsto X, \quad \mathbf{a}^{+} \mapsto Z, \quad \mathbf{a}^{-} \mapsto Z^{-1}(1 - X^2)$

q-Weyl algebra valued six vertex model

- L-operators L^A $(A = X, Z, O)$ [Kuniba-Matsuike-Y'22]
  - $L^A \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes F)$ $(A = X, Z)$ and $L^O \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes F_+)$

$$L^A(|i\rangle \otimes |j\rangle \otimes |k\rangle) = \sum_{a,b\in\{0,1\}} |a\rangle \otimes |b\rangle \otimes \pi_A((L^A)^{a,b}_{i,j})\,|k\rangle \quad (A = X, Z, O)$$



$(L^X)^{a,b}_{i,j} = (L^Z)^{a,b}_{i,j}$	r	s	twX	$-qtX$	Z	$Z^{-1}(rs - t^2wX^2)$
$(L^O)^{a,b}_{i,j}$	1	1	$\mu\mathbf{k}$	$-q\mu^{-1}\mathbf{k}$	$\mathbf{a}^+$	$\mathbf{a}^-$

r, s, t, w, μ: parameters

- Remark:
  - L^X for $(r, s, t, w) = (1, 1, \mu^{-1}, \mu^2)$ corresponds to L^O via the pullback.
  - L^Z doesn't have such a correspondence and behaves differently from L^O.
  - Slightly different but similar L^X was introduced in [Bazhanov-Mangazeev-Sergeev'10] but L^Z is new.

Family of $RLLL$ relations

- Our Problem:
  - Solve the following equation for R^{ABC} $(\mathrm{A}, \mathrm{B}, \mathrm{C} \in \{X, Z, O\})$:

$$L^{A}_{124} L^{B}_{135} L^{C}_{236} R^{ABC}_{456} = R^{ABC}_{456} L^{C}_{236} L^{B}_{135} L^{A}_{124}$$

r_4, s_4, t_4, w_4 or μ_4

r_5, s_5, t_5, w_5 or μ_5

r_6, s_6, t_6, w_6 or μ_6

  - Each L has different parameters depending on its tensor compoment.

Main result

- [Kuniba-Matsuike-Y'22]:
  - We solved $RLLL$ relations for the following ABCs.

ABC	feature	locally finiteness	$\sharp$(sector)
ZZZ	factorized	no	4
OZZ	${}_2\phi_1$	no	1
ZZO	${}_2\phi_1$	no	
ZOZ	${}_3\phi_2$-like	no	
OOZ	factorized	yes	1
ZOO	factorized	yes	
OZO	factorized	no	
OOO	${}_2\phi_1$	yes	1
XXZ	factorized	no	2
ZXX	factorized	no	
XZX	factorized	no	

  - For all cases, R^{ABC} are uniquely determined in each sector specified by appropriate parity conditions.
  - We obtained the explicit formulae for them, where their matrix elements are either factorized or expressed as q-hypergeometric series.

$RLLL$ relation for ZZZ

- Examples of $RLLL$ relation for ZZZ:

$$R(1\otimes X\otimes X)=(1\otimes X\otimes X)R,\quad R(X\otimes X\otimes 1)=(X\otimes X\otimes 1)R,$$
$$-r_1r_3R(1\otimes Z\otimes 1)=(qt_1t_3w_1X\otimes Z\otimes X-r_2Z\otimes 1\otimes Z)R,$$
$$R(-qt_1t_3w_3X\otimes Z\otimes X+s_2Z\otimes 1\otimes Z)=s_1s_3(1\otimes Z\otimes 1)R,$$
$$t_1R(X\otimes Z\otimes Z^{-1}(r_3s_3-t_3^2w_3X^2)+s_2t_3Z\otimes 1\otimes X)=s_3t_2(Z\otimes X\otimes 1)R,$$
$$R(t_3w_3Z^{-1}(r_1s_1-t_1^2w_1X^2)\otimes Z\otimes X+s_2t_1w_1X\otimes 1\otimes Z)=s_1t_2w_2(1\otimes X\otimes Z)R.$$

$$\pi_Z : X\left|m\right\rangle=\left|m-1\right\rangle,\quad Z\left|m\right\rangle=q^m\left|m\right\rangle$$

- Writing down actions of π_Z, we obtain recursion relations for R^{ZZZ}:

$$R^{a,b,c}_{i,j-1,k-1}=R^{a,b+1,c+1}_{i,j,k},\quad R^{a,b,c}_{i-1,j-1,k}=R^{a+1,b+1,c}_{i,j,k},$$
$$(q^{a+c}r_2-q^jr_1r_3)R^{a,b,c}_{i,j,k}=q^{1+b}t_1t_3w_1R^{a+1,b,c+1}_{i,j,k},$$
$$(q^{i+k}s_2-q^bs_1s_3)R^{a,b,c}_{i,j,k}=q^{1+j}t_1t_3w_3R^{a,b,c}_{i-1,j,k-1},$$
$$q^jr_3s_3t_1R^{a,b,c}_{i-1,j,k}-q^{j+2}t_1t_3^2w_3R^{a,b,c}_{i-1,j,k-2}+q^{i+k}s_2t_3R^{a,b,c}_{i,j,k-1}=q^{a+k}s_3t_2R^{a,b+1,c}_{i,j,k},$$
$$q^jr_1s_1t_3w_3R^{a,b,c}_{i,j,k-1}-q^{j+2}t_1^2t_3w_1w_3R^{a,b,c}_{i-2,j,k-1}+q^{i+k}s_2t_1w_1R^{a,b,c}_{i-1,j,k}=q^{c+i}s_1t_2w_2R^{a,b+1,c}_{i,j,k}$$

- <u>Fact</u>: Recursion relations for ZZZ consists of 4 disjoint sets, which are specified with the parity pair $(d_1,d_2)=(a+c-j,b-i-k)$.

$RLLL$ relation for ZZZ

- Thm: [Kuniba-Matsuike-Y'22]
  - $R^{ZZZ} \in \mathrm{End}(F^{\otimes 3})$ is uniquely determined in each sector and given by

$$R^{a,b,c}_{i,j,k} = \left(\frac{r_2}{t_1 t_3 w_1}\right)^{\frac{d_1}{2}} \left(\frac{s_2}{t_1 t_3 w_3}\right)^{\frac{d_2}{2}} \left(\frac{t_2}{s_1 t_3}\right)^{\frac{d_3}{2}} \left(\frac{t_2 w_2}{s_3 t_1 w_1}\right)^{\frac{d_4}{2}}$$

$$\times\, q^{\varphi} \frac{\Phi_{d_2}\left(\frac{s_1 s_3}{s_2}\right) \Phi_{d_3}\left(\frac{r_3 w_2}{s_3 w_1}\right) \Phi_{d_4}\left(\frac{r_1 w_3}{s_1 w_2}\right)}{\Phi_{-d_1}\left(\frac{q^2 r_1 r_3}{r_2}\right) \Phi_{d_3+d_4}\left(\frac{r_1 r_3 w_3}{s_1 s_3 w_1}\right)}, \qquad a,b,c,i,j,k \in \mathbb{Z}$$

$$\varphi = \frac{1}{4}\left((d_1 - d_2)(d_1 + d_2 + d_3 + d_4) + d_3 d_4\right) - d_1,$$

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} a + c - j \\ b - i - k \end{pmatrix}, \quad \begin{pmatrix} d_3 \\ d_4 \end{pmatrix} = \begin{pmatrix} -a - b + c + i + j - k \\ a - b - c - i + j + k \end{pmatrix}$$

$$\Phi_m(z) = \frac{1}{(z q^m; q^2)_\infty} \quad (m \in \mathbb{Z}),$$

- Features:
  - The matrix elements of R^{ZZZ} are factorized.
  - R^{ZZZ} is *not* locally finite.
  - There are 4 sectors specified with the parity pair (d_1, d_2).

$RLLL$ relation for OZZ

- Thm: [Kuniba-Matsuike-Y'22]
  - $R^{OZZ} \in \mathrm{End}(F_+ \otimes F \otimes F)$ is uniquely determined and given by

$$R^{a,b,c}_{i,j,k} = \theta(i \geq 0) \left(\frac{r_2}{r_3}\right)^a \left(\frac{s_3}{s_2}\right)^i \left(\frac{t_2 w_2}{\mu s_2}\right)^{-b+j} \left(-\frac{\mu t_3}{r_3}\right)^{-c+k} \frac{(z;q^2)_a}{(q^2;q^2)_a} q^{(a-b+j-1)c-(i-b+j-1)k-aj+bi}$$

$$\times {}_2\phi_1 \left({q^{-2i}, z^{-1}q^2 \atop z^{-1}q^{-2a+2}} ; q^2, yq^{2i+2j-2a-2b} \right). \qquad a, i \in \mathbb{Z}_{\geq 0},\ b, c, j, k \in \mathbb{Z}$$

$$\mu = \mu_4, \quad y = \frac{r_3 w_3}{\mu^2 s_3}, \quad z = q^{2k-2c+2} \frac{\mu^2 s_2}{r_2 w_2}$$

- Features:
  - The matrix elements of R^{OZZ} are expressed as q−hypergeometric series.
  - R^{OZZ} is *not* locally finite.
  - There is only 1 sector.

$RLLL$ relation for OOZ

- Thm: [Kuniba-Matsuike-Y'22]
  - $R^{OOZ} \in \mathrm{End}(F_+ \otimes F_+ \otimes F)$ is uniquely determined and non-trivial iff $\mu_1 / \mu_2 = q^d$ for d ∈ ℤ. In that case, it is given by

$$R(d)^{a,b,c}_{i,j,k} = \theta(e \in \mathbb{Z})\theta(\min(i,j) \geq 0)\delta^{a+b}_{i+j} \qquad a,b,i,j \in \mathbb{Z}_{\geq 0}, \ c,k \in \mathbb{Z}$$

$$\times\ s_3^i(\mu_2 t_3)^{-a}\left(\frac{\mu_2 s_3}{t_3 w_3}\right)^j\left(\frac{t_3^2 w_3}{r_3 s_3}\right)^e q^{cj-bk}\frac{(q^{2+2e-2j};q^2)_j(q^{2a+2};q^2)_{i-a}}{(q^2;q^2)_f(q^{2a-2e};q^2)_{e-a}}$$

$$e = \frac{1}{2}(a-c+j+k+d), \quad f = \frac{1}{2}(b+c+i-k-d)$$

- Features:
  - The matrix elements of R^{OOZ} are factorized.
  - R^{OOZ} is locally finite.
  - There is only 1 sector but R^{OOZ} is non-trivial if the parity of $2e$ is even.

$RLLL$ relation for OOO

- Thm: [Bazhanov-Sergeev'06]
  - $R^{OOO} \in \mathrm{End}(F_+^{\otimes 3})$ is uniquely determined and given by

$$(R^{OOO})^{a,b,c}_{i,j,k} = \delta^{a+b}_{i+j}\delta^{b+c}_{j+k}\left(\frac{\mu_3}{\mu_2}\right)^i\left(-\frac{\mu_1}{\mu_3}\right)^b\left(\frac{\mu_2}{\mu_1}\right)^k q^{ik+b(k-i+1)}\binom{a+b}{a}_{q^2}\,{}_2\phi_1\left(\begin{matrix}q^{-2b},q^{-2i}\\ q^{-2a-2b}\end{matrix};q^2,q^{-2c}\right)$$

$$a,b,c,i,j,k \in \mathbb{Z}_{\geq 0}$$

- Features:
  - The matrix elements of R^{OOO} are expressed as q−hypergeometric series.
  - R^{OOO} is locally finite.
  - There is only 1 sector.
  - R^{OOO} also satisfies the following tetrahedron equation:

$$R^{OOO}_{124}R^{OOO}_{135}R^{OOO}_{236}R^{OOO}_{456} = R^{OOO}_{456}R^{OOO}_{236}R^{OOO}_{135}R^{OOO}_{124}$$

  - R^{OOO} = intertwiner of irreps of quantum coordinate ring $A_q(A_2)$

$$R^{OOO}\circ(\pi_1\otimes\pi_2\otimes\pi_1(\Delta^{\mathrm{op}}(g))) = (\pi_2\otimes\pi_1\otimes\pi_2(\Delta(g)))\circ R^{OOO} \quad {}^\forall g \in A_q(A_2)$$

$$\pi_i : A_q(A_2) \to \mathrm{End}(F_+)$$

Outline

$RRRR$ equation as associaticity

- If we have $L_{124}L_{135}L_{236}R_{456} = R_{456}L_{236}L_{135}L_{124}$, we have

$$
\begin{aligned}
&R_{124}R_{135}R_{236}R_{456}\underline{L_{\alpha\beta6}L_{\alpha\gamma5}L_{\beta\gamma4}}L_{\alpha\delta3}L_{\beta\delta2}L_{\gamma\delta1} \\
&= R_{124}R_{135}R_{236}L_{\beta\gamma4}L_{\alpha\gamma5}\underline{L_{\alpha\beta6}L_{\alpha\delta3}L_{\beta\delta2}}L_{\gamma\delta1}R_{456} \\
&= R_{124}R_{135}L_{\beta\gamma4}\underline{L_{\alpha\gamma5}L_{\beta\delta2}}L_{\alpha\delta3}\underline{L_{\alpha\beta6}L_{\gamma\delta1}}R_{236}R_{456} \\
&= R_{124}R_{135}L_{\beta\gamma4}L_{\beta\delta2}\underline{L_{\alpha\gamma5}L_{\alpha\delta3}L_{\gamma\delta1}}L_{\alpha\beta6}R_{236}R_{456} \\
&= R_{124}\underline{L_{\beta\gamma4}L_{\beta\delta2}L_{\gamma\delta1}}L_{\alpha\delta3}L_{\alpha\gamma5}L_{\alpha\beta6}R_{135}R_{236}R_{456} \\
&= L_{\gamma\delta1}L_{\beta\delta2}\underline{L_{\beta\gamma4}L_{\alpha\delta3}}L_{\alpha\gamma5}L_{\alpha\beta6}R_{124}R_{135}R_{236}R_{456} \\
&= L_{\gamma\delta1}L_{\beta\delta2}L_{\alpha\delta3}L_{\beta\gamma4}L_{\alpha\gamma5}L_{\alpha\beta6}R_{124}R_{135}R_{236}R_{456}
\end{aligned}
$$

- $R_{456}R_{236}R_{135}R_{124}$ also gives an intertwiner for $\begin{cases} L_{\alpha\beta6}L_{\alpha\gamma5}L_{\beta\gamma4}L_{\alpha\delta3}L_{\beta\delta2}L_{\gamma\delta1} \\ L_{\gamma\delta1}L_{\beta\delta2}L_{\alpha\delta3}L_{\beta\gamma4}L_{\alpha\gamma5}L_{\alpha\beta6} \end{cases}$
- If they are irreducible and equivalent, we have

$$R_{124}R_{135}R_{236}R_{456} = R_{456}R_{236}R_{135}R_{124} \quad \text{(up to normalization)}$$

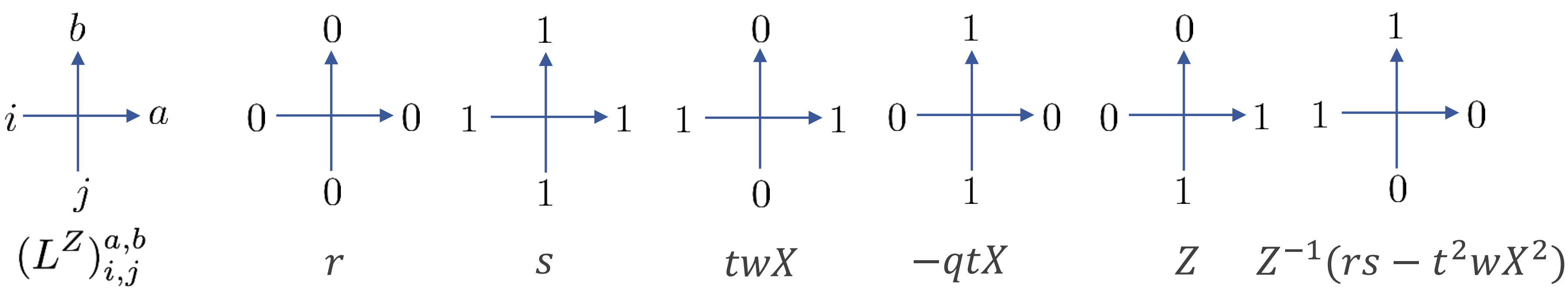
$RRRR$ equations for R^{ABC}

- For our $RLLL$ relations, we expect the following $RRRR$ equation holds:

$$R_{124}^{ABD} R_{135}^{ACE} R_{236}^{BCF} R_{456}^{DEF} = R_{456}^{DEF} R_{236}^{BCF} R_{135}^{ACE} R_{124}^{ABD}$$

$$A, B, C, D, E, F \in \{X, Z, O\}$$

- Remark:
  - Each tensor component is assigned with different parameters.
    - e.g. If $A = B = C = D = E = F = Z$, this depends on r_i, s_i, t_i, w_i $(i = 1, \ldots, 6)$.
  - R^{ABC}s except for $ABC = OOZ, ZOO, OOO$ are not locally finite, so the convergence of $RRRR$ equation is non-trivial for such cases.
  - L^Z is not irreducible because $(L^Z)_{i,j}^{a,b}$ does not include X^{-1}.



$$\pi_Z : X\,|m\rangle = |m-1\rangle\,, \quad Z\,|m\rangle = q^m\,|m\rangle$$

$RRRR$ equations for R^{ABC}

- <u>Conjecture</u>: [Kuniba-Matsuike-Y'22]
  - The following $RRRR$ equations are valid:

$$
\begin{aligned}
R^{OOO}_{456}R^{OOO}_{236}R^{ZOO}_{135}R^{ZOO}_{124} &= R^{ZOO}_{124}R^{ZOO}_{135}R^{OOO}_{236}R^{OOO}_{456}\\
R^{ZOO}_{456}R^{OOO}_{236}R^{OOO}_{135}R^{OOZ}_{124} &= R^{OOZ}_{124}R^{OOO}_{135}R^{OOO}_{236}R^{ZOO}_{456}\\
R^{OOZ}_{456}R^{OOZ}_{236}R^{OOO}_{135}R^{OOO}_{124} &= R^{OOO}_{124}R^{OOO}_{135}R^{OOZ}_{236}R^{OOZ}_{456}\\
R^{OOZ}_{456}R^{OOZ}_{236}R^{ZOO}_{135}R^{ZOO}_{124} &= R^{ZOO}_{124}R^{ZOO}_{135}R^{OOZ}_{236}R^{OOZ}_{456}.
\end{aligned}
$$

$$
\begin{aligned}
R^{OOO}_{456}R^{ZOO}_{236}R^{OOO}_{135}R^{OZO}_{124} &= R^{OZO}_{124}R^{OOO}_{135}R^{ZOO}_{236}R^{OOO}_{456}\\
R^{OZO}_{456}R^{OOO}_{236}R^{OOZ}_{135}R^{OOO}_{124} &= R^{OOO}_{124}R^{OOZ}_{135}R^{OOO}_{236}R^{OZO}_{456}\\
R^{OOO}_{456}R^{ZOO}_{236}R^{ZOO}_{135}R^{ZZO}_{124} &= R^{ZZO}_{124}R^{ZOO}_{135}R^{ZOO}_{236}R^{OOO}_{456}\\
R^{ZOO}_{456}R^{OOO}_{236}R^{ZOO}_{135}R^{ZOZ}_{124} &= R^{ZOZ}_{124}R^{ZOO}_{135}R^{OOO}_{236}R^{ZOO}_{456}\\
R^{ZOO}_{456}R^{ZOO}_{236}R^{OOO}_{135}R^{OZZ}_{124} &= R^{OZZ}_{124}R^{OOO}_{135}R^{ZOO}_{236}R^{ZOO}_{456}\\
R^{ZZO}_{456}R^{OOO}_{236}R^{OOZ}_{135}R^{OOZ}_{124} &= R^{OOZ}_{124}R^{OOZ}_{135}R^{OOO}_{236}R^{ZZO}_{456}\\
R^{ZOZ}_{456}R^{OOZ}_{236}R^{OOO}_{135}R^{OOZ}_{124} &= R^{OOZ}_{124}R^{OOO}_{135}R^{OOZ}_{236}R^{ZOZ}_{456}\\
R^{OZZ}_{456}R^{OOZ}_{236}R^{OOZ}_{135}R^{OOO}_{124} &= R^{OOO}_{124}R^{OOZ}_{135}R^{OOZ}_{236}R^{OZZ}_{456}\\
R^{ZOO}_{456}R^{ZOO}_{236}R^{ZOO}_{135}R^{ZZZ}_{124} &= R^{ZZZ}_{124}R^{ZOO}_{135}R^{ZOO}_{236}R^{ZOO}_{456},\\
R^{ZZZ}_{456}R^{OOZ}_{236}R^{OOZ}_{135}R^{OOZ}_{124} &= R^{OOZ}_{124}R^{OOZ}_{135}R^{OOZ}_{236}R^{ZZZ}_{456}.
\end{aligned}
$$

$$
\begin{aligned}
R^{OOO}_{456}R^{OZO}_{236}R^{OZO}_{135}R^{OOO}_{124} &= R^{OOO}_{124}R^{OZO}_{135}R^{OZO}_{236}R^{OOO}_{456}\\
R^{OOO}_{456}R^{OZO}_{236}R^{ZZO}_{135}R^{ZOO}_{124} &= R^{ZOO}_{124}R^{ZZO}_{135}R^{OZO}_{236}R^{OOO}_{456}\\
R^{OZO}_{456}R^{OOO}_{236}R^{ZOZ}_{135}R^{ZOO}_{124} &= R^{ZOO}_{124}R^{ZOZ}_{135}R^{OOO}_{236}R^{OZO}_{456}\\
R^{OOO}_{456}R^{ZZO}_{236}R^{OZO}_{135}R^{OZO}_{124} &= R^{OZO}_{124}R^{OZO}_{135}R^{ZZO}_{236}R^{OOO}_{456}\\
R^{OOZ}_{456}R^{ZOZ}_{236}R^{OOO}_{135}R^{OZO}_{124} &= R^{OZO}_{124}R^{OOO}_{135}R^{ZOZ}_{236}R^{OOZ}_{456}\\
R^{ZOO}_{456}R^{OZO}_{236}R^{OZO}_{135}R^{OOZ}_{124} &= R^{OOZ}_{124}R^{OZO}_{135}R^{OZO}_{236}R^{ZOO}_{456}\\
R^{OZO}_{456}R^{OZO}_{236}R^{OZZ}_{135}R^{OOO}_{124} &= R^{OOO}_{124}R^{OZZ}_{135}R^{OZO}_{236}R^{OZO}_{456}\\
R^{OOZ}_{456}R^{OZZ}_{236}R^{OZO}_{135}R^{OOO}_{124} &= R^{OOO}_{124}R^{OZO}_{135}R^{OZZ}_{236}R^{OOZ}_{456}\\
R^{OZO}_{456}R^{OZO}_{236}R^{ZZZ}_{135}R^{ZOO}_{124} &= R^{ZOO}_{124}R^{ZZZ}_{135}R^{OZO}_{236}R^{OZO}_{456},\\
R^{OOZ}_{456}R^{ZZZ}_{236}R^{OZO}_{135}R^{OZO}_{124} &= R^{OZO}_{124}R^{OZO}_{135}R^{ZZZ}_{236}R^{OOZ}_{456}.
\end{aligned}
$$

<u>Rermark</u>: Each equation is checked for over 10000 outer lines by computer.

R^{ZZZ} as intertwiner of $A_q(A_2)$

- Proposition: [Kuniba-Matsuike-Y'22]
  - $R^{ZZZ} \in \mathrm{End}(F^{\otimes 3})$ satisfies the following intertwining relation of the quantum coordinate ring $A_q(A_2)$:

$$R^{ZZZ} \circ (\pi_1 \otimes \pi_2 \otimes \pi_1(\Delta^{\mathrm{op}}(g))) = (\pi_2 \otimes \pi_1 \otimes \pi_2(\Delta(g))) \circ R^{ZZZ} \quad {}^{\forall} g \in A_q(A_2)$$

  - $\pi_i = \pi_Z \circ \varrho_i$, where ϱ_1 and ϱ_2 are respectively given by t_{ij}: generators of $A_q(A_2)$

$$\begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} \mapsto \begin{pmatrix} Z^{-1}(u_1 - g_1 h_1 X^2) & g_1 X & 0 \\ -q h_1 X & Z & 0 \\ 0 & 0 & u_1^{-1} \end{pmatrix}, \begin{pmatrix} u_2^{-1} & 0 & 0 \\ 0 & Z^{-1}(u_2 - g_2 h_2 X^2) & g_2 X \\ 0 & -q h_2 X & Z \end{pmatrix}$$

  - π_is are not irreducible. $\pi_Z : X|m\rangle = |m-1\rangle, \quad Z|m\rangle = q^m |m\rangle$
  - Identification of parameters is done as follows:

$$u_1 = u_2 (=: u) \qquad g_1 h_1 = g_2 h_2 (=: p)$$

$$\frac{r_1}{t_1} = \frac{r_2}{t_2}, \quad \frac{s_2}{t_2} = \frac{s_3}{t_3}, \quad \frac{r_2}{r_1 r_3} = u, \quad \frac{s_1 s_3}{s_2} = u^2, \quad \frac{t_1^2 w_1}{r_1 s_1} = \frac{t_2^2 w_2}{r_2 s_2} = \frac{t_3^2 w_3}{r_3 s_3} = \frac{p}{u}.$$

Root of unity

- If we specialize q to a root of unity, the Fock spaces F, F_+ become finite dimensional. If we can formulate R^{ABC} in such cases…

- Extension of family of $RRRR$ equations:
  - Getting over its non locally finiteness, we obtain more family of $RRRR$ equations.

- Connection with physical models:
  - Finite dimensional solutions to tetrahedron equations are quite important because they can be used to construct tractable 3D transfer matrices.
  - [Bazhanov-Mangazeev-Sergeev'10] introduced $(L^X)'$ which is slightly different from L^X and solved $(R^{XXX})'$ at N-th root of unity. They found

    $(R^{XXX})' \cong$ Bazhanov-Baxter model
    (*spectral parameter dependent* solution to tetrahedron equation)

    ⬇ reduction [Bazhanov-Baxter'92]

    generalized chiral Potts model
    $\cong$ 2D R matrices associated with $U_q(A_{n-1}^{(1)})$ at root of unity

Other comments

- Boundary integrability in 3D:

$R(LLL) = (LLL)R$ (Yang-Baxter equation up to conjugation) ➡ RRRR=RRRR (Tetrahedron equation)

$K(LGLG) = (GLGL)K$ (reflection equation up to conjugation) ➡ RKRRKKR=RKKRRKR (3D reflection equation)

  - a q-Weyl algebra version of [Kuniba-Pasquier'18], [Kuniba-Okado-Y'19]?

- Reduction to 2D:
  - Generally, infinitely many solutions to the Yang-Baxter equation are obtained from one solution to the tetrahedron equation.
  - For R^{OOO}, they are identified with R matrices associated with

reduction	R matrices [Kuniba-Okado'14]
by trace	$U_q(A_{n-1}^{(1)})$, symmetric tensor rep.
by boundary vector	$U_q(D_{n+1}^{(2)})$, $U_q(A_{2n}^{(2)})$, $U_q(C_n^{(1)})$, Fock rep.

Other comments

- Characterization in terms of PBW bases:
  - Let us consider the transition matrix γ for PBW bases of quantum enveloping algebra $U_q(A_2)$:

$$e_2^{(a)} e_{12}^{(b)} e_1^{(c)} = \sum_{i,j,k} \gamma_{i,j,k}^{a,b,c} e_1^{(k)} e_{21}^{(j)} e_2^{(i)} \quad \cdots (*) \qquad i,j,k,a,b,c \in \mathbb{Z}_{\geq 0}$$

  - $e_i^{(a)}$: divided power given by $e_i^{(a)} = e_i^a/[a]!$
  - Theorem: [Sergeev'07], [Kuniba-Okado-Yamada'13]

$$\gamma_{i,j,k}^{a,b,c} = (R^{OOO})_{i,j,k}^{a,b,c}$$

  - Can we formulate R^{ABC} in this context?

Summary

- We considered three kinds of L-operators $L^X, \mathrm{L}^\mathrm{Z}, \mathrm{L}^\mathrm{O}$ and $RLLL$ relations which they satisfy. They can be regarded as q-Oscillator or q-Weyl algebra valued six vertex models.
- We solved these $RLLL$ relations and obtained explicit formulae for R^{ABC}. For all cases, R^{ABC} are uniquely determined in each sector specified by appropriate parity conditions and their matrix elements are either factorized or expressed as q-hypergeometric series.
- By computer experiments, we conjectured $RRRR$ equations for R^{ABC}. This is motivated by earlier results about representation theoretic origin of R^{OOO}.
- We found R^{ZZZ} satisfies an intertwining relation for *reducible* representations of $A_q(A_2)$.