3D reflection maps from tetrahedron maps

Mathematical Society of Japan Autumn Meeting
Chiba University@2021/09/14

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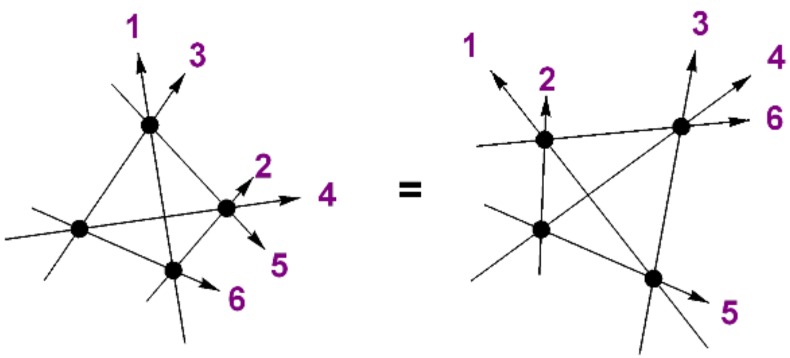
Based on: AY, Math. Phys. Anal. Geom. **24** 21 (2021)

Integrability in 3D

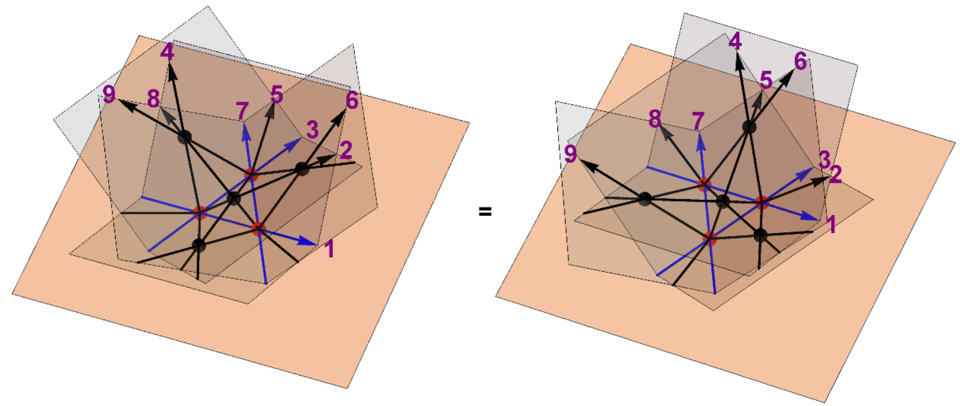
- Tetrahedron equation



$$\mathbf{R}_{245}\mathbf{R}_{135}\mathbf{R}_{126}\mathbf{R}_{346} = \mathbf{R}_{346}\mathbf{R}_{126}\mathbf{R}_{135}\mathbf{R}_{245}$$

[Zamolodchikov80]

- 3D reflection equation



$$\mathbf{R}_{489}\mathbf{J}_{3579}\mathbf{R}_{269}\mathbf{R}_{258}\mathbf{J}_{1678}\mathbf{J}_{1234}\mathbf{R}_{456} = \mathbf{R}_{456}\mathbf{J}_{1234}\mathbf{J}_{1678}\mathbf{R}_{258}\mathbf{R}_{269}\mathbf{J}_{3579}\mathbf{R}_{489}$$

[Isaev-Kulish97]

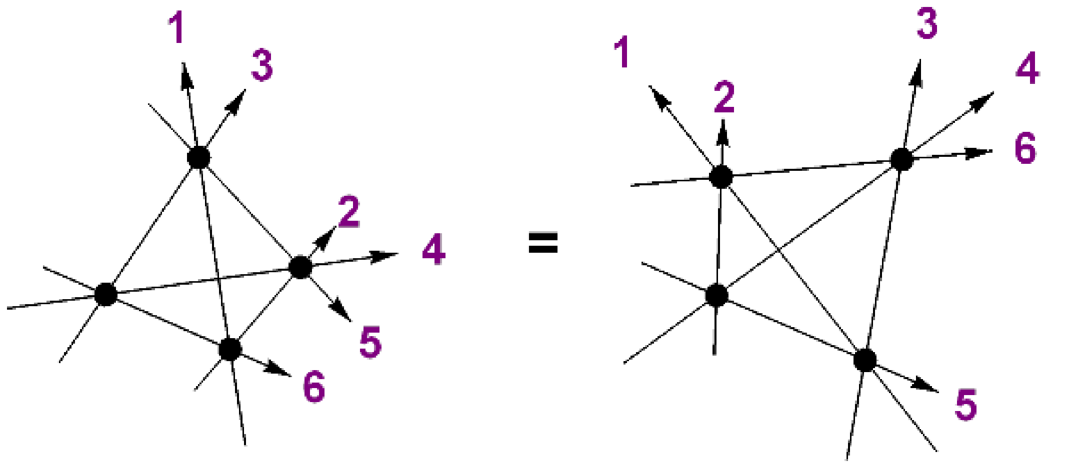
- Tetrahedron and 3D reflection equation are conditions for factorization of string scattering amplitude in 2+1D.

	Bulk	Boundary
2D	Yang-Baxter eq.	Reflection eq.
3D	Tetrahedron eq.	3D Reflection eq.

Aim & Motivation

- Several tetrahedron maps are known although less systematically than Yang-Baxter maps.
  - In the context of the local YBE [Sergeev98]
  - Transition maps of Lusztig's parametrizations of the canonical basis of $U_q(A_2)$ and their geometric liftings [Kuniba-Okado12] $\cdots (1)$
  - By using some KP tau functions [Kassotakis-Nieszporski-Papageorgiou-Tongas19]
- On the other hand, there are very few known 3D reflection maps.
  - Transition maps of Lusztig's parametrizations of the canonical basis of $U_q(B_2)$ and $U_q(C_2)$, and their geometric liftings [Kuniba-Okado12] $\cdots (2)$
- Aim: Obtain 3D reflection maps from known tetrahedron maps
- Motivation:
  - Some 2D reflection maps are constructed from known Yang-Baxter maps. [Caudrelier-Zhang14], [Kuniba-Okado19]
  - A relation between (1) and (2) is known associated with folding the Dynkin diagram of A_3 into one of B_2. [Berenstein-Zelevinsky01], [Lusztig11]
    → Let's generalize this!

Tetrahedron maps



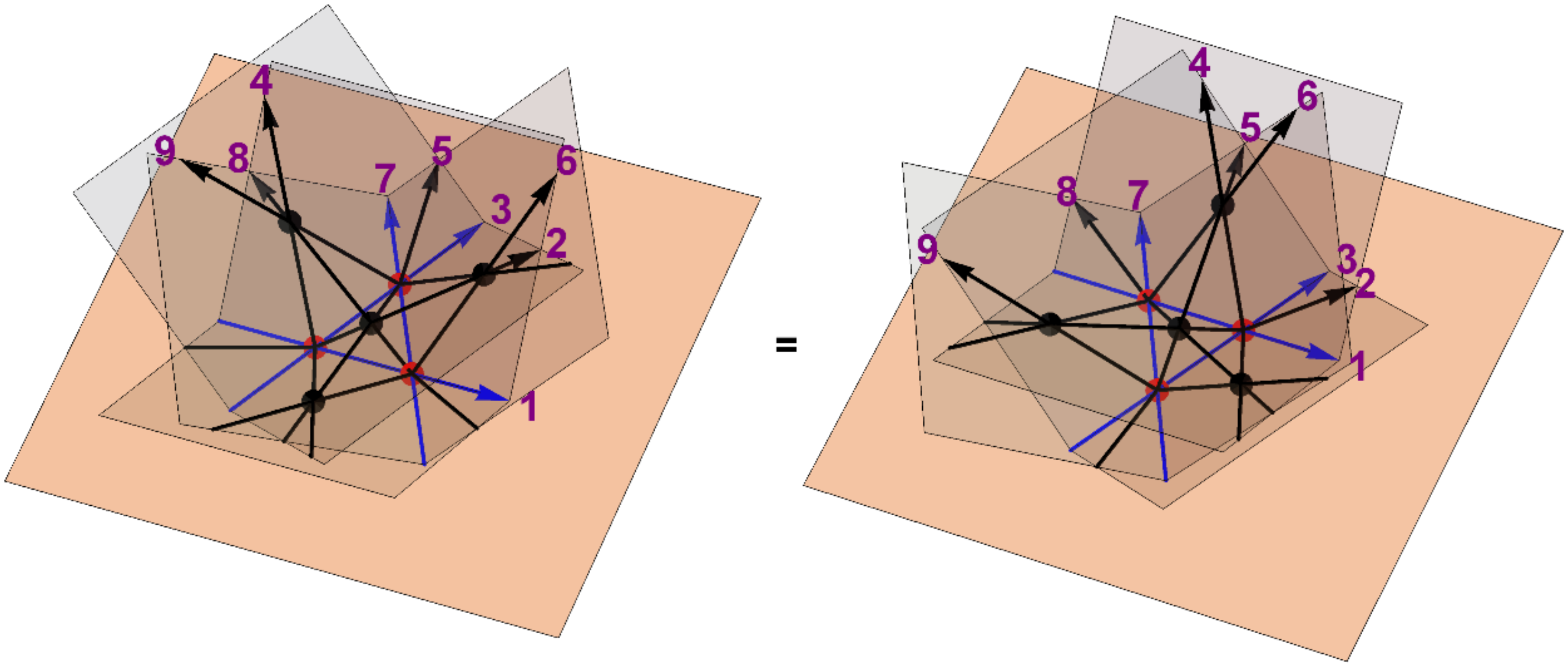
- <u>Definition</u>:
  - Let $\mathbf{R}: X^3 \to X^3$ (X: an arbitrary set) denote a map.
  - We call $\mathbf{R}$ *tetrahedron map* if it satisfies the tetrahedron equation on X^6:

$$\mathbf{R}_{245}\mathbf{R}_{135}\mathbf{R}_{126}\mathbf{R}_{346} = \mathbf{R}_{346}\mathbf{R}_{126}\mathbf{R}_{135}\mathbf{R}_{245} (=: \mathbf{T}_{123456}) \qquad \cdots (*)$$

  - We call $\mathbf{T}$ the *tetrahedral composite* of the tetrahedron map $\mathbf{R}$.
  - We call $\mathbf{R}$ *involutive* if $\mathbf{R}^2 = \mathrm{id}$ and *symmetric* if $\mathbf{R}_{123} = \mathbf{R}_{321}$.

- <u>Remark</u>:
  - For involutive and symmetric tetrahedron maps, $(*)$ corresponds to the usual tetrahedron equation.

3D reflection maps



- <u>Definition</u>:
  - Let $\mathbf{J}: X^4 \to X^4$ denote a map.
  - We set a tetrahedron map by $\mathbf{R}: X^3 \to X^3$.
  - We call $\mathbf{J}$ *3D reflection map* if it satisfies the 3D reflection equation on X^9:

$$\mathbf{R}_{489}\mathbf{J}_{3579}\mathbf{R}_{269}\mathbf{R}_{258}\mathbf{J}_{1678}\mathbf{J}_{1234}\mathbf{R}_{456} = \mathbf{R}_{456}\mathbf{J}_{1234}\mathbf{J}_{1678}\mathbf{R}_{258}\mathbf{R}_{269}\mathbf{J}_{3579}\mathbf{R}_{489}$$

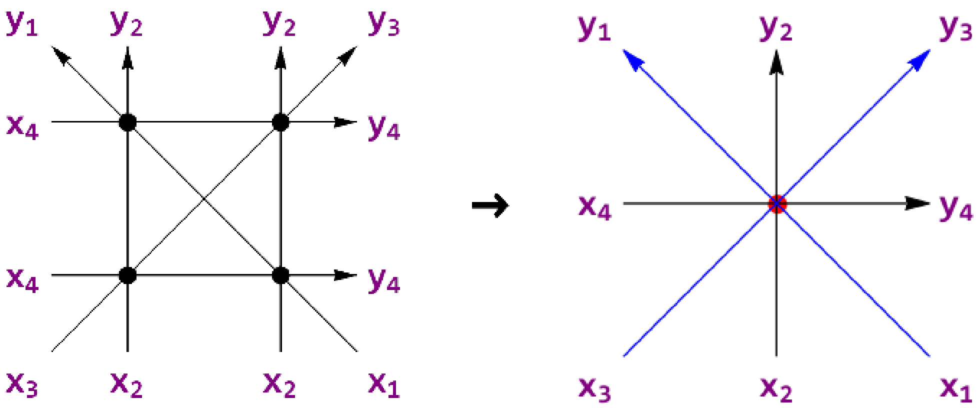
Boundarization

- We set the subset of X^6 by $Y = \{(x_1, \cdots, x_6) \mid x_2 = x_3, x_5 = x_6\}$.
  - We set $\phi: X^4 \to Y$ by $\phi(x_1, x_2, x_3, x_4) = (x_1, x_2, x_2, x_3, x_4, x_4)$ (embedding)
  - We set $\varphi: Y \to X^4$ by $\varphi(x_1, x_2, x_2, x_3, x_4, x_4) = (x_1, x_2, x_3, x_4)$ (projection)
- <u>Definition</u>:
  - Let $\mathbf{R}: X^3 \to X^3$ denote a tetrahedron map and $\mathbf{T}$ its tetrahedral composite.
  - We call $\mathbf{R}$ *boundarizable* if the following condition is satisfied:

$$x \in Y \;\Rightarrow\; \mathbf{T}(x) \in Y$$

  - In that case, we define the *boundarization* $\mathbf{J}: X^4 \to X^4$ of $\mathbf{R}$ by

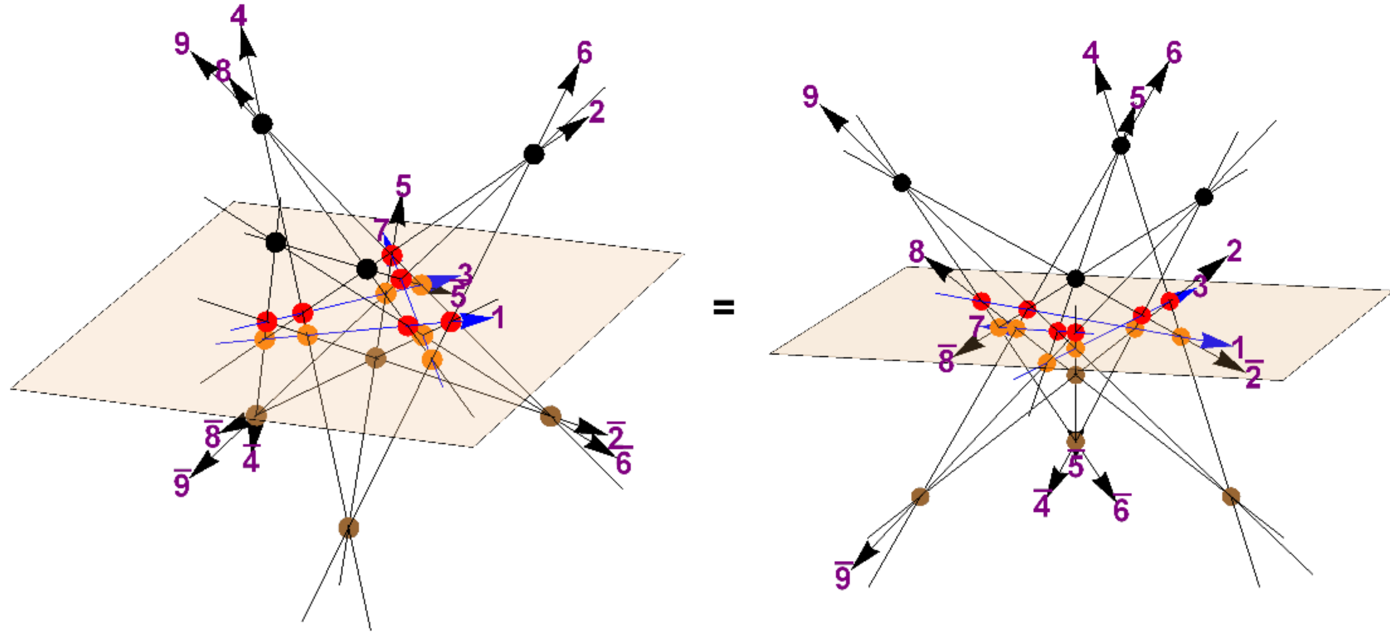
$$\mathbf{J}(\mathbf{x}) = \varphi(\mathbf{T}(\phi(\mathbf{x})))$$



$$y_i = \varphi(\mathbf{T}(\phi(\mathbf{x})))_i$$

Main theorem

- Theorem:
  - Let $\mathbf{R}: X^3 \to X^3$ denote an involutive, symmetric and boundarizable tetrahedron map, and $\mathbf{J}: X^4 \to X^4$ its boundarization.
  - Then they satisfy 3D reflection equation.
- Sketch of Proof:
  - Cut the following identiy on X^{15} into half:



Example: birational transition map

- We set $\mathbf{R}\colon \mathbb{R}^3_{>0} \to \mathbb{R}^3_{>0}$ by

$$\mathbf{R} : (x_1, x_2, x_3) \mapsto (\tilde{x_1}, \tilde{x_2}, \tilde{x_3}) = \left(\frac{x_1 x_2}{x_1 + x_3}, x_1 + x_3, \frac{x_2 x_3}{x_1 + x_3} \right)$$

  - This map is characterized as the transition map of parametrizations of the positive part of SL_3: [Lusztig94]

$$G_1(x_3)G_2(x_2)G_1(x_1) = G_2(\tilde{x_1})G_1(\tilde{x_2})G_2(\tilde{x_3}) \qquad G_i(x) = 1 + xE_{i,i+1}$$

  - We can verify $\mathbf{R}$ is the involutive, symmetric and boundarizable tetrahedron map.

- The associated 3D reflection map **J**: $\mathbb{R}^4_{>0} \to \mathbb{R}^4_{>0}$ is calculated as:

$$\mathbf{J} : (x_1, x_2, x_3, x_4) \mapsto \left(\frac{x_1 x_2^2 x_3}{y_1}, \frac{y_1}{y_2}, \frac{y_2^2}{y_1}, \frac{x_2 x_3 x_4}{y_2} \right)$$

$$y_1 = x_1(x_2 + x_4)^2 + x_3 x_4^2, \quad y_2 = x_1(x_2 + x_4) + x_3 x_4$$

  - This map is exactly the transition map of parametrizations of the positive part of SP_4, which is a consequence from folding the Dynkin diagram of A_3 into one of C_2. [Berenstein-Zelevinsky01], [Lusztig11]

Concluding remarks

- Summary:
  - We present a method for obtaining 3D reflection maps by using known tetrahedron maps, which is an analog of the results in 2D.
  - Our method is a kind of generalization of the relation by Berenstein and Zelevinsky and gives 3D interpretation to their relation.
  - By applying our method to known tetrahedron maps, we obtain several 3D reflection maps which include new solutions.
- Remark:
  - Our theorem can be extended to *inhomogeneous* cases, that is, the case tetrahedron maps are defined on direct product of different sets.