Tetrahedron equation from PBW bases of the nilpotent subalgebra of quantum superalgebras

Mathsci Freshman Seminar@2021/02/13

Akihito Yoneyama
Institute of Physics, University of Tokyo, Komaba

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Outline

Yang-Baxter equation and R matrices

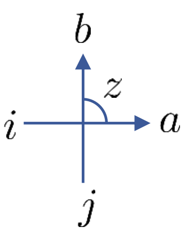
- Matrix equation on $V_1 \otimes V_2 \otimes V_3$ (V_i: linear space)

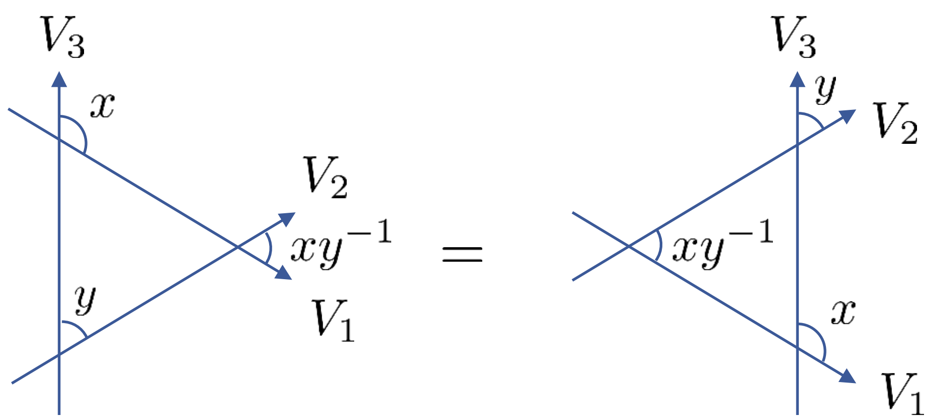
$$R_{12}(xy^{-1})R_{13}(x)R_{23}(y) = R_{23}(y)R_{13}(x)R_{12}(xy^{-1})$$

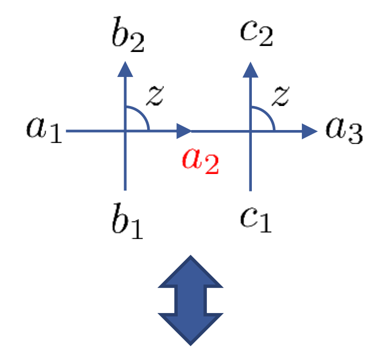
  - $R(z) : W_1 \otimes W_2 \to W_1 \otimes W_2$ (W_i: linear space)
  - $R_{ij}(z)$ acts non-trivially only on $V_i \otimes V_j$.

- Graphical notation:

$$R(z)\,|i\rangle \otimes |j\rangle = \sum_{a,b} R(z)_{i,j}^{a,b}\,|a\rangle \otimes |b\rangle$$

$R(z)_{i,j}^{a,b} =$ 





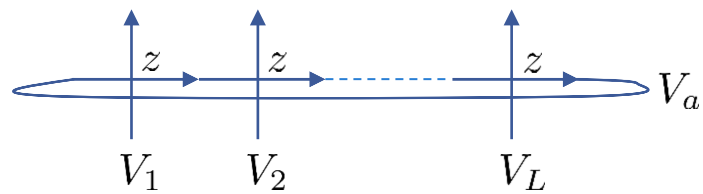
$$\sum_{a_2} R(z)_{a_1,b_1}^{a_2,b_2} R(z)_{a_2,c_1}^{a_3,c_2}$$

Commuting transfer matrix

- From Yang-Baxter equation, we have the following:
- <u>Proposition</u>:
  - We set the *transfer matrix* $\tau(z)\colon \mathrm{V}^{\otimes L} \to \mathrm{V}^{\otimes L}$ by

$V_i = V$

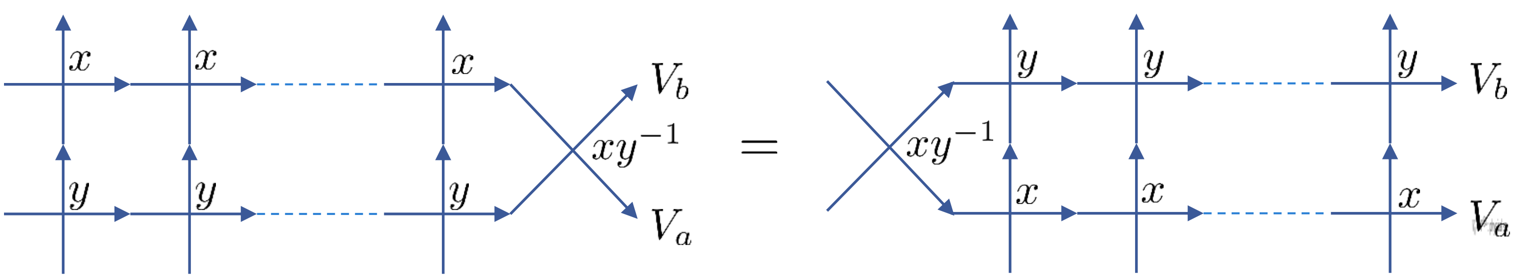
$$\tau(z) := \mathrm{Tr}_a \left(R_{a1}(z) R_{a2}(z) \ldots R_{aL}(z) \right) =$$



  - Then, we have

$$[\tau(x), \tau(y)] = 0 \quad ({}^{\forall} x, y \in \mathbb{C})$$

- <u>Proof</u>:
  - For the following eq, multiply $R_{ab}(xy^{-1})^{-1}$ and take trace on $V_a \otimes V_b$:



1D integrable hamiltonian

- A lot of families of integrable one-dimensional quantum spin chains are constructed via commutativty of the transfer matrix.
- The following $\mathcal{H}$ gives one-dimensional quantum spin chain

$$\mathcal{H} := \frac{\mathrm{d}}{\mathrm{d}z}(\log\tau(z))\Big|_{z=1}$$

  - This has a nearest-neighbor interaction (physical):

$$\mathcal{H} = \sum_i \mathcal{H}_{i,i+1} \qquad \mathcal{H}_{i,i+1} : V_i \otimes V_{i+1} \to V_i \otimes V_{i+1}$$

- Integrability: the transfer matrix gives conserved quantities:

$$[\tau(x), \tau(y)] = 0 \quad ({}^\forall x, y \in \mathbb{C}) \quad \Rightarrow \quad [\mathcal{H}, \tau(y)] = 0 \quad {}^\forall y \in \mathbb{C}$$

$$\tau(y) = \underline{\tau(1)} + \underline{\tau'(1)}y + \cdots$$

  - All eigenvalues of $\tau(z)$ are obtained by Bethe ansatz for large classes of R matrices.

Reps of quantum algebra & R matrices

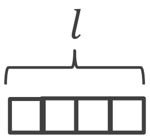
- R matrices are systematically (infinitely many) constructed via irreps of Drinfeld-Jimbo quantum affine algebra $U_q(g)$.
  - g: affine Lie algebra $A_n^{(1)},\ B_n^{(1)},\ C_n^{(1)},\ D_n^{(1)}, \dots$
  - $U_q(g)$ is one-parametric generalization of g, and reduces to g with $q \to 1$
- $(V_1, \pi_z^{(1)}), (V_2, \pi_z^{(2)})$: irreps of $U_q(g)$ $\quad$ z: *spectral parameter*
  - Example: Kirillov-Reshetikhin module
- If $\pi_x^{(1)} \otimes \pi_y^{(2)}, \pi_y^{(2)} \otimes \pi_x^{(1)}$ are irreducible, a linear map $\check{R}(z)$ given by

$$(\pi_y^{(2)} \otimes \pi_x^{(1)})(\Delta(g))\check{R}(x/y) = \check{R}(x/y)(\pi_x^{(1)} \otimes \pi_y^{(2)})(\Delta(g)) \quad {}^\forall g \in U_q(X)$$

  is uniquely determined by Schur's Lemma.
- $R(z) = P\check{R}(z)$ gives a solution to the Yang-Baxter equation.
  - ``linearization" method for the Yang-Baxter equation. $\quad P\,|i\rangle \otimes |j\rangle = |j\rangle \otimes |i\rangle$
- Point: characterization $\neq$ explicit formula

Explicit formula for R matrices

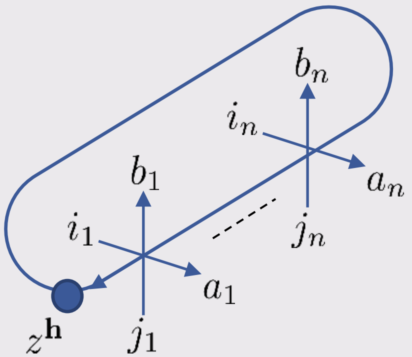
- For some $U_q(g)$ and their representations, there exist explicit formulae for R matrices from *three dimensional integrability*.
- <u>Example</u>:



  - $U_q(g) = U_q(A_{n-1}^{(1)})$
  - $\pi_z^{(1)} = \pi_z^{(2)} =$ arbitrary (l-th) symmetric tensor representation

  - <u>Theorem</u>:

$$R(z)_{\boldsymbol{i},\boldsymbol{j}}^{\boldsymbol{a},\boldsymbol{b}} = \mathrm{Tr}_F(z^{\mathbf{h}}\mathcal{R}_{i_1,j_1}^{a_1,b_1}\cdots\mathcal{R}_{i_n,j_n}^{a_n,b_n}) =$$



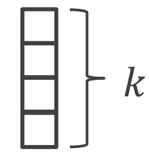
satisfies the intertwining relation

$$(\pi_x^{(1)} \otimes \pi_y^{(2)})(\Delta^{\mathrm{op}}(g))R(x/y) = R(x/y)(\pi_x^{(1)} \otimes \pi_y^{(2)})(\Delta(g)) \quad {}^{\forall}g \in U_q(X)$$

  - $\mathcal{R}$ is the solution to the tetrahedron equation called $3DR$.

Explicit formula for R matrices

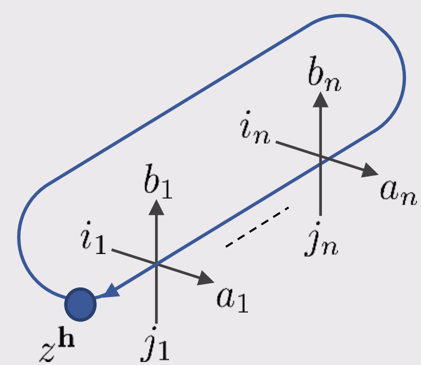
- For some $U_q(g)$ and their representations, there exist explicit formulae for R matrices from *three dimensional integrability*.
- Example:
  - $U_q(g) = U_q(A^{(1)}_{n-1})$
  - $\pi^{(1)}_z = \pi^{(2)}_z =$ arbitrary (k-th) fundamental representation



$1 \le k \le n-1$

- Theorem:

$$R(z)^{\boldsymbol{a},\boldsymbol{b}}_{\boldsymbol{i},\boldsymbol{j}} = \mathrm{Tr}_F(z^{\mathbf{h}} \mathcal{L}^{a_1,b_1}_{i_1,j_1} \cdots \mathcal{L}^{a_n,b_n}_{i_n,j_n}) =$$



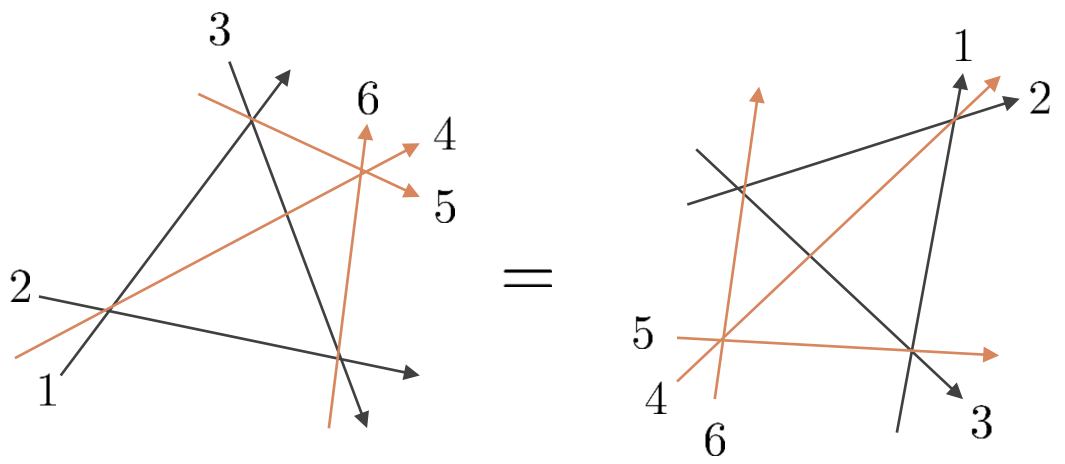
satisfies the intertwining relation

$$(\pi^{(1)}_x \otimes \pi^{(2)}_y)(\Delta^{\mathrm{op}}(g))R(x/y) = R(x/y)(\pi^{(1)}_x \otimes \pi^{(2)}_y)(\Delta(g)) \quad {}^\forall g \in U_q(X)$$

- $\mathcal{L}$ is the solution to the tetrahedron equation called $3DL$.

Outline

Tetrahedron equation



- Matrix equation on $V_1 \otimes \cdots \otimes V_6$ (V_i: linear space)

$$\mathcal{R}_{124}\mathcal{R}_{135}\mathcal{R}_{236}\mathcal{R}_{456} = \mathcal{R}_{456}\mathcal{R}_{236}\mathcal{R}_{135}\mathcal{R}_{124}$$

  - R_{ijk} acts non-trivially only on $V_i \otimes V_j \otimes V_k$.
  - Tetrahedron equation = Yang-Baxter equation *up to conjugation*

- Unlike Yang-Baxter equation, a few families of solutions are known.

- We focus on solutions on the Fock spaces.

boson Fock: $F = \bigoplus_{m=0,1,2,\cdots} \mathbb{C}\,|m\rangle$

fermi Fock: $V = \bigoplus_{m=0,1} \mathbb{C}u_m$

Solutions to tetrahedron equation: 3DR

- Set $\mathcal{R} \in \mathrm{End}\left(F^{\otimes 3}\right)$ by [Kapranov-Voevodsky94]

$$\mathcal{R}\,|i\rangle \otimes |j\rangle \otimes |k\rangle = \textstyle\sum_{a,b,c} \mathcal{R}_{ijk}^{abc}\,|a\rangle \otimes |b\rangle \otimes |c\rangle$$

$$\mathcal{R}_{i,j,k}^{a,b,c} = \delta_{i+j}^{a+b}\delta_{j+k}^{b+c} \sum_{\lambda,\mu\geq 0,\lambda+\mu=b} (-1)^{\lambda} q^{i(c-j)+(k+1)\lambda+\mu(\mu-k)} \frac{(q^2)_{c+\mu}}{(q^2)_c} \binom{i}{\mu}_{q^2} \binom{j}{\lambda}_{q^2}$$

Here

$$(q)_k = \textstyle\prod_{l=1}^{k}(1-q^l) \qquad \binom{a}{b}_q = \frac{(q)_a}{(q)_b (q)_{a-b}}$$

- The 3DR satisfies the following tetrahedron equation:

$$\mathcal{R}_{124}\mathcal{R}_{135}\mathcal{R}_{236}\mathcal{R}_{456} = \mathcal{R}_{456}\mathcal{R}_{236}\mathcal{R}_{135}\mathcal{R}_{124}$$

- 3DR = intertwiner of irreps of quantum coordinate ring $A_q(A_2)$

$$\mathcal{R} \circ \pi_1 \otimes \pi_2 \otimes \pi_1(\Delta^{\mathrm{op}}(g)) = \pi_2 \otimes \pi_1 \otimes \pi_2(\Delta(g)) \circ \mathcal{R} \quad {}^{\forall} g \in A_q(A_2)$$

$$\pi_i : A_q(A_2) \to \mathrm{End}(F)$$

  - This gives a linearization method for tetrahedron equation.
  - ``121" and ``212" are associated with the longest element of Weyl group.

Solutions to tetrahedron equation: 3DL

- Set $\mathcal{L} \in \mathrm{End}\,(V \otimes V \otimes F)$ by [Bazhanov-Sergeev06]

$$\mathcal{L}(u_i \otimes u_j \otimes |k\rangle) = \sum_{a,b\in\{0,1\},c\in\mathbb{Z}_{\geq 0}} \mathcal{L}^{a,b,c}_{i,j,k} u_a \otimes u_b \otimes |c\rangle$$

$$\mathcal{L}^{0,0,c}_{0,0,k} = \mathcal{L}^{1,1,c}_{1,1,k} = \delta_{k,c}, \quad \mathcal{L}^{0,1,c}_{0,1,k} = -\delta_{k,c}q^{k+1}, \quad \mathcal{L}^{1,0,c}_{1,0,k} = \delta_{k,c}q^{k},$$
$$\mathcal{L}^{0,1,c}_{1,0,k} = \delta_{k-1,c}(1-q^{2k}), \quad \mathcal{L}^{1,0,c}_{0,1,k} = \delta_{k+1,c}$$

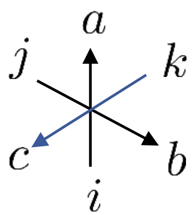
- The 3DL satisfies the following tetrahedron equation:

$$\mathcal{L}_{124}\mathcal{L}_{135}\mathcal{L}_{236}\mathcal{R}_{456} = \mathcal{R}_{456}\mathcal{L}_{236}\mathcal{L}_{135}\mathcal{L}_{124} \qquad \cdots(*)$$

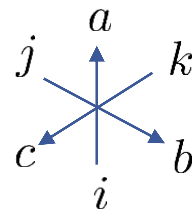
- BS obtained the 3DL by ansatz so that the tetrahedron equation of $(*)$ type has a non-trivial solution, and solved $(*)$ for the 3DR.
  - They gives an alternative derivation for the 3DR.
  - Algebraic origins of the 3DL has been still unclear.

Reduction to Yang-Baxter equation

- We use the following graphical notations:

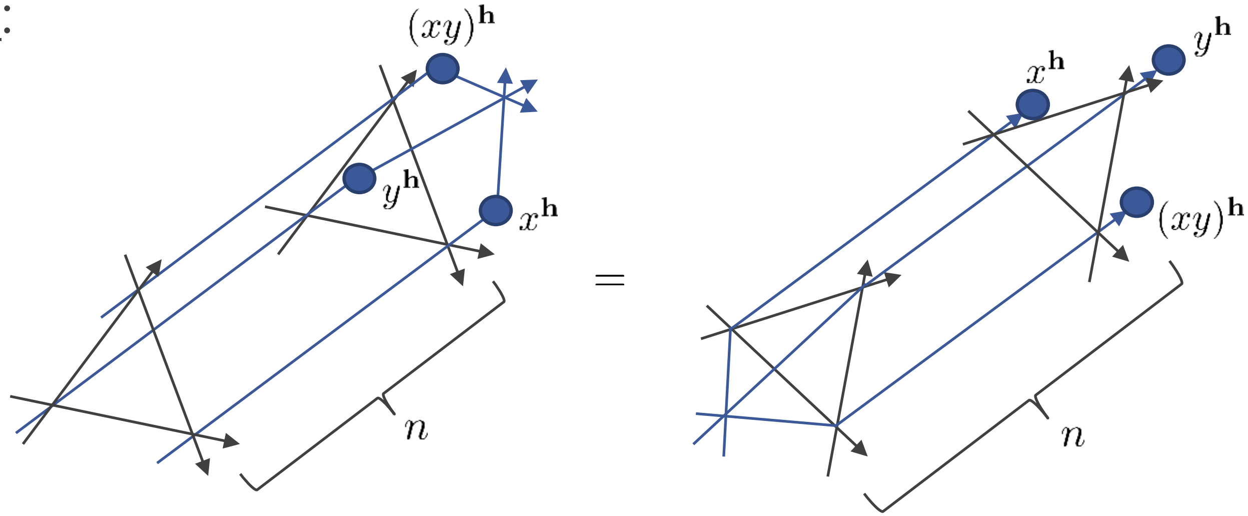


$= \mathcal{R}^{a,b,c}_{i,j,k}$ or $\mathcal{L}^{a,b,c}_{i,j,k}$



$= \mathcal{R}^{a,b,c}_{i,j,k}$

- Fact:

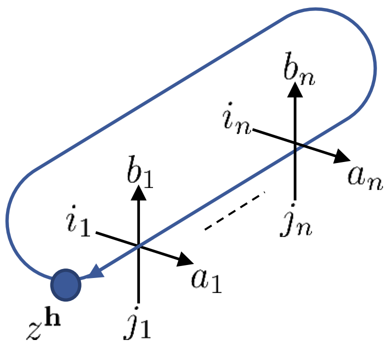


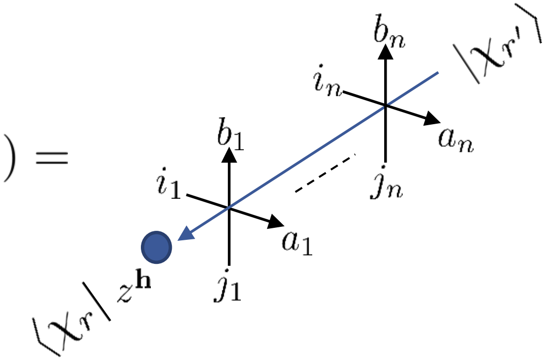
$^\forall x, y \in \mathbb{C}$

  - $\mathbf{h}$: number counting operator of the Fock spaces $\quad \mathbf{h}\,|m\rangle = m\,|m\rangle$

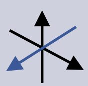
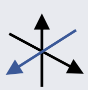
- By multiplying $\mathcal{R}^{-1}$ and taking trace on the auxiliary spaces, we obtain the Yang-Baxter equation.

Family of matrix product solutions

$R^{\mathrm{tr}}(z) =$ 

$R^{(r,r')}(z) =$ 

- These solutions are characterized as the R matrices associated with some quantum affine algebras. [Kuniba-Okado-Sergeev15]

	$R^{\mathrm{tr}}(z)$	$R^{(r,r')}(z)$
 = 3DR	$U_q(A_{n-1}^{(1)})$ symmetric tensor rep.	$U_q(D_{n+1}^{(2)})$, $U_q(A_{2n}^{(2)})$, $U_q(C_n^{(1)})$ Fock rep.
 = 3DL	$U_q(A_{n-1}^{(1)})$ fundamental rep.	$U_q(D_{n+1}^{(2)})$, $U_q(B_n^{(1)})$, $U_q(D_n^{(1)})$ spin rep.

- Moreover, by mixing uses of the 3DR & 3DL, we also obtain the R matrices associated with *generalized quantum groups*.

Motivation & KOY theorem

- Motivation:
  - Why do the 3DR & 3DL lead to such similar results, although they have different origins?
- We study transition matrices of PBW bases of $U_q^+(sl(m|n))$ motivated by the KOY theorem which holds for non-super cases.
- Theorem: [Kuniba-Okado-Yamada13] (Rough Statement)
  - g: arbitrary finite-dimensional simple Lie algebra
  - Φ: intertwiner of irreducible representations of $A_q(g)$
  - γ: transition matrix of PBW bases of the nilpotent subalgebra of $U_q(g)$
  - Then, we have $\Phi = \gamma$.
- For ○——○, the 3DR gives the transition matrix for $U_q(A_2)$.
- One of our result is that the 3DL is exactly the transition matrix for the quantum superalgebra associated with ○——⊗.

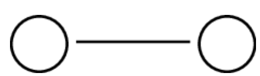
Outline

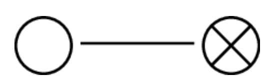
Root data of Lie superalgebra of type A

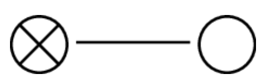
- We only consider rank 2 cases.
- Lie superalgebra $sl(m|n)$ (m+n=3)
  - Simple roots: $\Pi = \{\alpha_1, \alpha_2\}$
  - Parity: p: $\Pi \to \{0,1\}$
  - Positive roots: $\Phi = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$ $\quad p(\alpha_1 + \alpha_2) = p(\alpha_1) + p(\alpha_2)$ (mod2)
  - Even & odd parts of positive roots: $\Phi = \Phi_{\mathrm{even}} \cup \Phi_{\mathrm{odd}}$
- Dynkin diagram for $sl(m|n)$ (m+n=3)
  - Prepare connected 2 dots and decorate the i-th dot by

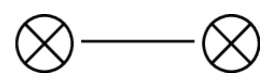
$$\bigcirc \text{ for } \alpha_i \in \Phi_{\mathrm{even}} \ , \ \otimes \text{ for } \alpha_i \in \Phi_{\mathrm{odd}}$$

  - All diagrams are listed as follows

(1) 

(2) 

(3) 

(4) 

PBW bases for $U_q^+(sl(m|n))$ of rank 2

- $U_q^+ = U_q^+(sl(m|n))$: nilpotent subalgebra of $U_q(sl(m|n)$ (m+n=3)
  - Generator: $\{e_1, e_2\}$
  - If $\alpha_i \in \Phi_{\mathrm{even}}$, $e_i^2 e_{3-i} - (q+q^{-1})e_i e_{3-i} e_i + e_{3-i} e_i^2 = 0$
  - If $\alpha_i \in \Phi_{\mathrm{odd}}$, $e_i^2 = 0$
- PBW bases for U_q^+
  - We set $e_{ij} = e_i e_j - q e_j e_i$. Then, the following B_1 & B_2 give bases of U_q^+:

$$B_1 = \left\{ e_1^{(a_{\alpha_1})} e_{21}^{(a_{\alpha_1+\alpha_2})} e_2^{(a_{\alpha_2})} \mid a_\alpha \in \mathbb{Z}_{\geq 0} \ (\alpha \in \Phi_{\mathrm{even}}),\ a_\alpha \in \{0,1\} \ (\alpha \in \Phi_{\mathrm{odd}}) \right\}$$

$$B_2 = \left\{ e_2^{(a_{\alpha_2})} e_{12}^{(a_{\alpha_1+\alpha_2})} e_1^{(a_{\alpha_1})} \mid a_\alpha \in \mathbb{Z}_{\geq 0} \ (\alpha \in \Phi_{\mathrm{even}}),\ a_\alpha \in \{0,1\} \ (\alpha \in \Phi_{\mathrm{odd}}) \right\}$$

  - $e_i^{(a)}$: divided power given by $e_i^{(a)} = e_i^a/[a]!$ $\quad [m]! = \prod_{k=1}^m [k]$
- Transition matrix γ $\quad [k] = \frac{q^k - q^{-k}}{q - q^{-1}}$

$$e_2^{(a)} e_{12}^{(b)} e_1^{(c)} = \sum_{i,j,k} \gamma_{i,j,k}^{a,b,c} e_1^{(k)} e_{21}^{(j)} e_2^{(i)}$$

The case $\bigcirc\!\!-\!\!\!-\!\!\!-\!\!\bigcirc$

- Root system

$$\Phi_{\mathrm{even}} = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\} \quad \Phi_{\mathrm{odd}} = \{\}$$

- Transition matrix

$$e_2^{(a)} e_{12}^{(b)} e_1^{(c)} = \sum_{i,j,k} \gamma_{i,j,k}^{a,b,c} e_1^{(k)} e_{21}^{(j)} e_2^{(i)} \quad \cdots (*) \qquad i,j,k,a,b,c \in \mathbb{Z}_{\geq 0}$$

- <u>Theorem</u>: [Kuniba-Okado-Yamada13]

$$\gamma_{i,j,k}^{a,b,c} = \mathcal{R}_{i,j,k}^{a,b,c}$$

- <u>Example</u>: For $(a, b, c) = (0,1,1)$, $(*)$ becomes

$$e_{12}e_1 = \mathcal{R}_{0,1,1}^{0,1,1} e_1 e_{21} + \mathcal{R}_{1,0,2}^{0,1,1} \frac{e_1^2}{q+q^{-1}} e_2$$

$$-qe_2e_1^2 + (1+q^2)e_1e_2e_1 - qe_1^2e_2 = 0 \qquad \because \mathcal{R}_{0,1,1}^{0,1,1} = -q^2,\ \mathcal{R}_{1,0,2}^{0,1,1} = 1 - q^4$$

This is a relation of $\bigcirc\!\!-\!\!\!-\!\!\!-\!\!\bigcirc$.

The case ○——⊗

- Root system

$$\Phi_{\text{even}} = \{\alpha_1\} \quad \Phi_{\text{odd}} = \{\alpha_2, \alpha_1 + \alpha_2\}$$

- Transition matrix

$$e_2^{(a)} e_{12}^{(b)} e_1^{(c)} = \sum_{i,j,k} \gamma_{i,j,k}^{a,b,c} e_1^{(k)} e_{21}^{(j)} e_2^{(i)} \quad \cdots (*) \qquad \begin{array}{l} k, c \in \mathbb{Z}_{\geq 0} \\ i, j, a, b \in \{0, 1\} \end{array}$$

- Theorem: [Y20]

$$\gamma_{i,j,k}^{a,b,c} = \mathcal{L}_{i,j,k}^{a,b,c}$$

- Example: For $(a, b, c) = (0,1,1)$, $(*)$ becomes

$$e_{12} e_1 = \mathcal{L}_{0,1,1}^{0,1,1} e_1 e_{21} + \mathcal{L}_{1,0,2}^{0,1,1} \frac{e_1^2}{q + q^{-1}} e_2$$

$$-q e_2 e_1^2 + (1 + q^2) e_1 e_2 e_1 - q e_1^2 e_2 = 0 \qquad \because \mathcal{L}_{0,1,1}^{0,1,1} = -q^2,\ \mathcal{L}_{1,0,2}^{0,1,1} = 1 - q^4$$

This is a relation of ○——⊗.

The case $\bigotimes$——$\bigotimes$

- Root system

$$\Phi_{\mathrm{even}} = \{\alpha_1 + \alpha_2\} \quad \Phi_{\mathrm{odd}} = \{\alpha_1, \alpha_2\}$$

- Transition matrix

$$e_2^{(a)} e_{12}^{(b)} e_1^{(c)} = \sum_{i,j,k} \gamma_{i,j,k}^{a,b,c} e_1^{(k)} e_{21}^{(j)} e_2^{(i)} \quad \cdots (*) \qquad \begin{array}{l} j, b \in \mathbb{Z}_{\geq 0} \\ i, k, a, c \in \{0, 1\} \end{array}$$

- <u>Theorem</u>: [Y20]

$$\gamma_{i,j,k}^{a,b,c} = \mathcal{N}_{i,j,k}^{a,b,c}$$

  - Here, we define $\mathcal{N} \in \mathrm{End}\,(V \otimes F \otimes V)$ as follows:

$$\mathcal{N}(u_i \otimes |j\rangle \otimes u_k) = \sum_{a,c \in \{0,1\}, b \in \mathbb{Z}_{\geq 0}} \mathcal{N}_{i,j,k}^{a,b,c} u_a \otimes |b\rangle \otimes u_c$$

$$\mathcal{N}_{0,j,0}^{0,b,0} = \delta_{j,b} q^j, \quad \mathcal{N}_{1,j,1}^{1,b,1} = -\delta_{j,b} q^{j+1}, \quad \mathcal{N}_{0,j,1}^{0,b,1} = \mathcal{N}_{1,j,0}^{1,b,0} = \delta_{j,b},$$

$$\mathcal{N}_{1,j,1}^{0,b,0} = \delta_{j+1,b} q^j (1 - q^2), \quad \mathcal{N}_{0,j,0}^{1,b,1} = \delta_{j-1,b} [j]_q,$$

Remark for rank 3

- Transition matrices γ for higher rank cases are constructed as compositions of ones for rank 2.
- By constructing γ for rank 3 in two ways, we obtain the tetrahedron equation.

$$e_3^{(o_1)} e_{23}^{(o_2)} e_2^{(o_3)} e_{123}^{(o_4)} e_{12}^{(o_5)} e_1^{(o_6)} = \sum_{i_1,i_2,i_3,i_4,i_5,i_6} \gamma^{o_1,o_2,o_3,o_4,o_5,o_6}_{i_1,i_2,i_3,i_4,i_5,i_6} e_1^{(i_6)} e_{21}^{(i_5)} e_{321}^{(i_4)} e_2^{(i_3)} e_{32}^{(i_2)} e_3^{(i_1)}$$

- For 

$$\mathcal{R}_{123}\mathcal{R}_{145}\mathcal{R}_{246}\mathcal{R}_{356} = \mathcal{R}_{356}\mathcal{R}_{246}\mathcal{R}_{145}\mathcal{R}_{123}$$

- For 

$$\mathcal{L}_{123}\mathcal{L}_{145}\mathcal{L}_{246}\mathcal{R}_{356} = \mathcal{R}_{356}\mathcal{L}_{246}\mathcal{L}_{145}\mathcal{L}_{123}$$

- For  (new solution to the tetrahedron equation)

$$\mathcal{N}(q^{-1})_{123}\mathcal{N}(q^{-1})_{145}\mathcal{R}_{246}\mathcal{L}_{356} = \mathcal{L}_{356}\mathcal{R}_{246}\mathcal{N}(q^{-1})_{145}\mathcal{N}(q^{-1})_{123}$$

Concluding remarks

- <u>Remark</u>:
  1. Type B cases give new solutions to the 3D reflection equations, which describes boundary integrability in three dimensions.
  2. The crystal limit for $\mathcal{L}, \mathcal{N}$ gives a super analog of transition maps of Lusztig's parametrizations of the canonical basis of quantum algebras.

- <u>Summary</u>:
  1. The 3DL is characterized as the transition matrix for $\bigcirc\!-\!\!\!-\!\!\!-\!\otimes$.
  2. A new solution to the tetrahedron equation the 3DN is obtained by considering the transition matrix for $\otimes\!-\!\!\!-\!\!\!-\!\otimes$.