Matrix product solutions to reflection equation and coideal subalgebras of $U_q(A^{(1)}_{n-1})$

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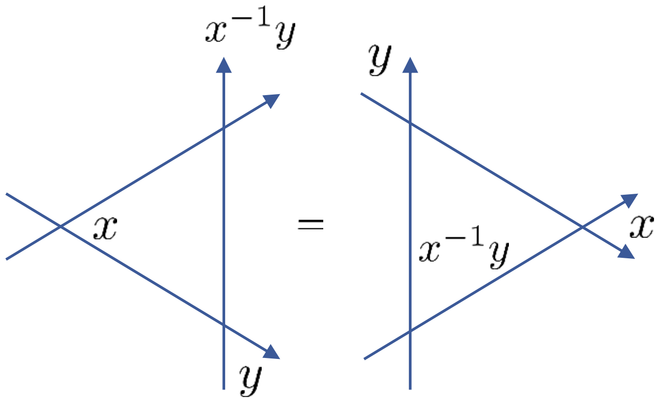
Based on: A.Kuniba, and M.Okado, and AY, arXiv: 1812.03767

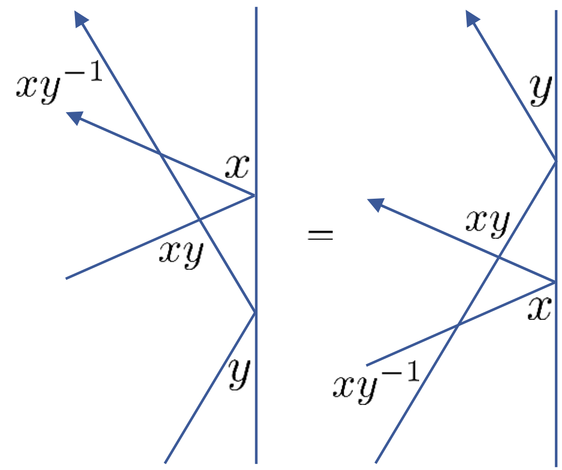
YBE and RE in 2d integrability

- Yang-Baxter equation



$$R_{12}(x)R_{13}(y)R_{23}(x^{-1}y)$$
$$= R_{23}(x^{-1}y)R_{13}(y)R_{12}(x)$$

- Reflection equation



$$R_{\mathbf{12}}(xy^{-1})K_{\mathbf{2}}(x)R_{\mathbf{21}}(xy)K_{\mathbf{1}}(y)$$
$$= K_{\mathbf{1}}(y)R_{\mathbf{12}}(xy)K_{\mathbf{2}}(x)R_{\mathbf{21}}(xy^{-1})$$

- <u>Point</u>: characterization vs explicit formula
- <u>Strategy</u>: *3d structure* → an explicit formula

Outline

Quantized coordinate ring $A_p(G)$

- the Hopf algebra dual to $U_p(g)$ [Drinfeld,87],[RTF,90] etc…
- Defined on arbitrary simply connected simple Lie group G
- Simplest Example: $A_p(SL_2)$

Recall

$$SL_2 = \left\{ \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \in \mathrm{End}(\mathbb{C}^2) \middle| [t_{ij}, t_{kl}] = 0,\ t_{11}t_{22} - t_{12}t_{21} = 1 \right\}$$

$A_p(SL_2) = \langle t_{11}, t_{12}, t_{21}, t_{22} \rangle$ with the relations

$$t_{11}t_{12} = pt_{12}t_{11},\ t_{11}t_{21} = pt_{21}t_{11},\ t_{12}t_{22} = pt_{22}t_{12},\ t_{21}t_{22} = pt_{22}t_{21}$$

$$[t_{12}, t_{21}] = 0,\ [t_{11}, t_{22}] = (p - p^{-1})t_{21}t_{12} \qquad t_{11}t_{22} - pt_{12}t_{21} = 1$$

becomes Hopf algebra with coproduct $\Delta(t_{ij}) = \sum_k t_{ik} \otimes t_{kj}$

Representation of $A_p(G)$

- The <u>irreducible reps</u> of $A_p(G)$ are classified in [Soibelman,91]

  $\overset{1:1}{\longleftrightarrow}$ elements of the Weyl group $W(g)$

- Let p-boson $\langle \mathbf{1}, \mathbf{a}^+, \mathbf{a}^-, \mathbf{h} \rangle$ with the relations

  $$\mathbf{k} = p^{\mathbf{h}},\ \mathbf{k}\mathbf{a}^{\pm} = p^{\pm}\mathbf{a}^{\pm}\mathbf{k},\ \mathbf{a}^-\mathbf{a}^+ = \mathbf{1} - p^2\mathbf{k}^2,\ \mathbf{a}^+\mathbf{a}^- = \mathbf{1} - \mathbf{k}^2$$

  and its Fock space $F_p = \oplus_{m\geq 0}\mathbb{C}\,|m\rangle$ with

  $$\mathbf{k}\,|m\rangle = p^m\,|m\rangle\,,\ \mathbf{a}^+\,|m\rangle = |m+1\rangle\,,\ \mathbf{a}^-\,|m\rangle = (1-p^{2m})\,|m-1\rangle$$

- <u>Theorem</u> [Soibelman,91]
  - All the irreps of $A_p(G)$ are given as follows:

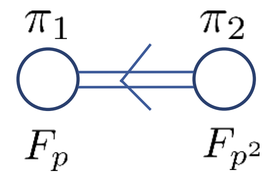
    $s_i \in W(g)$ (simple reflection) $\longleftrightarrow$ $^{\exists}\pi_i : A_p(G) \to \mathrm{End}(F)$

    $s_{i_1}\cdots s_{i_r}$ (reduced expression) $\longleftrightarrow$ $\pi_{i_1}\otimes\cdots\otimes\pi_{i_r}$

  - If $s_{i_1}\cdots s_{i_r} = s_{j_1}\cdots s_{j_{r'}}$, then $\pi_{i_1}\otimes\cdots\otimes\pi_{i_r} \simeq \pi_{j_1}\otimes\cdots\otimes\pi_{j_r}$

Example: $A_p(SP_4)$

- $A_p(SP_4) = \langle t_{ij} \rangle_{i,j=1}^{4}$



$$\begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ t_{41} & t_{42} & t_{43} & t_{44} \end{pmatrix} \mapsto \overset{\pi_1}{\begin{pmatrix} \mathbf{a}^- & \mathbf{k} & 0 & 0 \\ -p\mathbf{k} & \mathbf{a}^+ & 0 & 0 \\ 0 & 0 & \mathbf{a}^- & -\mathbf{k} \\ 0 & 0 & p\mathbf{k} & \mathbf{a}^+ \end{pmatrix}}, \overset{\pi_2}{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathbf{A}^- & \mathbf{K} & 0 \\ 0 & -p^2\mathbf{K} & \mathbf{A}^+ & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

$\langle \mathbf{a}^{\pm}, \mathbf{k} \rangle$:$p$-boson

$\langle \mathbf{A}^{\pm}, \mathbf{K} \rangle$:$p^2$-boson

- $W(sp_4) = \langle s_1, s_2 \rangle \qquad s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1$

  ⇒ $\pi_1 \otimes \pi_2 \otimes \pi_1 \otimes \pi_2 \simeq \pi_2 \otimes \pi_1 \otimes \pi_2 \otimes \pi_1$

- By Schur's Lemma $^\exists \Phi : F_p \otimes F_{p^2} \otimes F_p \otimes F_{p^2} \to F_{p^2} \otimes F_p \otimes F_{p^2} \otimes F_p$

  $$\pi_2 \otimes \pi_1 \otimes \pi_2 \otimes \pi_1(\Delta(t_{ij})) \circ \Phi = \Phi \circ \pi_1 \otimes \pi_2 \otimes \pi_1 \otimes \pi_2(\Delta(t_{ij}))$$

- Set $\mathcal{K} = \Phi \circ \sigma \in \mathrm{End}\left(F_{p^2} \otimes F_p \otimes F_{p^2} \otimes F_p\right) \quad \sigma(x \otimes y \otimes z \otimes w) = w \otimes z \otimes y \otimes x$

  $$\pi_2 \otimes \pi_1 \otimes \pi_2 \otimes \pi_1(\Delta(t_{ij})) \circ \mathcal{K} = \mathcal{K} \circ \pi_2 \otimes \pi_1 \otimes \pi_2 \otimes \pi_1(\Delta^{\mathrm{op}}(t_{ij}))$$

  Here, $\Delta^{\mathrm{op}}(t_{ij}) = \sigma \Delta(t_{ij}) \sigma$

3dK

- Denote its component as $\mathcal{K}|i,j,k,l\rangle = \sum_{a,b,c,d} \mathcal{K}^{abcd}_{ijkl}|a,b,c,d\rangle$
- Prop [Kuniba-Okado,12] $\quad |i,j,k,l\rangle := |i\rangle \otimes |j\rangle \otimes |k\rangle \otimes |l\rangle$
  - With the normalization $\mathcal{K}|0,0,0,0\rangle = |0,0,0,0\rangle$

$$\mathcal{K}^{a,b,c,d}_{i,j,k,l} = \frac{(p^4)_i}{(p^4)_a} \sum_{\alpha,\beta,\gamma} \frac{(-1)^{\alpha+\gamma}}{(p^4)_{c-\beta}} p^{\phi_1} \mathcal{K}^{i,j+k-\alpha-\beta-\gamma,0,l+k-\alpha-\beta-\gamma}_{a,b+c-\alpha-\beta-\gamma,0,c+d-\alpha-\beta-\gamma} \left\{ \begin{array}{c} k, c-\beta, j+k-\alpha-\beta, k+l-\alpha-\beta \\ \alpha,\beta,\gamma, b-\alpha, d-\alpha, k-\alpha-\beta, c-\beta-\gamma \end{array} \right\}$$

$$\mathcal{K}^{a,b,0,d}_{i,j,0,l} = \sum_{\lambda} (-1)^{b+\lambda} \frac{(p^4)_{a+\lambda}}{(p^4)_a} p^{\phi_2} \left\{ \begin{array}{c} j,l \\ \lambda, l-\lambda, b-\lambda, j-b+\lambda \end{array} \right\}$$

$$\phi_1 = \alpha(\alpha+2c-2\beta-1) + (2\beta-c)(b+c+d) + \gamma(\gamma-1) - k(j+k+l)$$

$$\phi_2 = (i+a+1)(b+l-2\lambda) + b - l$$

Here,

$$(p)_k = \prod_{l=1}^{k}(1-p^l) \qquad \left\{ \begin{array}{c} i_1,\cdots,i_r \\ i_1,\cdots,i_s \end{array} \right\} = \begin{cases} \frac{\prod_{k=1}^{r}(p^2)_{i_k}}{\prod_{k=1}^{s}(p^2)_{j_k}} & \forall i_k, j_k \in \mathbb{Z}_{\geq 0} \\ 0 & \text{otherwise} \end{cases}$$

  - $\mathcal{K}^{a,b,c,d}_{i,j,k,l} \in p^{\eta}\mathbb{Z}[p^2] \quad \eta \equiv jl + bd \mod 2$
  - $\mathcal{K}^{-1} = \mathcal{K}$
  - $[(xy^{-1})^{\mathbf{h}_1} x^{\mathbf{h}_2} (xy)^{\mathbf{h}_3} y^{\mathbf{h}_4}, \mathcal{K}] = 0$

$$\mathbf{h}_1 = \mathbf{h} \otimes 1 \otimes 1 \otimes 1$$

$$\mathbf{h}|m\rangle = m|m\rangle$$

Quantized reflection equation

- Recall the intertwining relations for 3dK

$$\pi_2\otimes\pi_1\otimes\pi_2\otimes\pi_1(\Delta(t_{ij}))\circ\mathcal{K} = \mathcal{K}\circ\pi_2\otimes\pi_1\otimes\pi_2\otimes\pi_1(\Delta^{\mathrm{op}}(t_{ij})) \quad (i,j=1,\cdots,4)$$

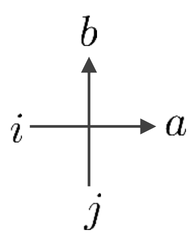
⟺ RE up to conjugation =: Quantized RE [Kuniba-Pasquier,18]

$$L_{123}G_{24}L_{215}G_{16}\mathcal{K}_{3456} = \mathcal{K}_{3456}G_{16}L_{125}G_{24}L_{213} \in \mathrm{End}(\overset{1}{V}\otimes\overset{2}{V}\otimes\overset{3}{F_{p^2}}\otimes\overset{4}{F_p}\otimes\overset{5}{F_{p^2}}\otimes\overset{6}{F_p})$$

$$V=\mathbb{C}v_0\oplus\mathbb{C}v_1$$

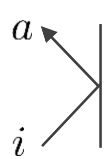
- $L_{i,j}^{a,b}$ and G_i^a act on the auxiliary Fock spaces. $i,j,a,b\in\{0,1\}$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p\mathbf{K} & \mathbf{A}^- & 0 \\ 0 & \mathbf{A}^+ & -p\mathbf{K} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{End}\left(V\otimes V\otimes F_{p^2}\right)$$

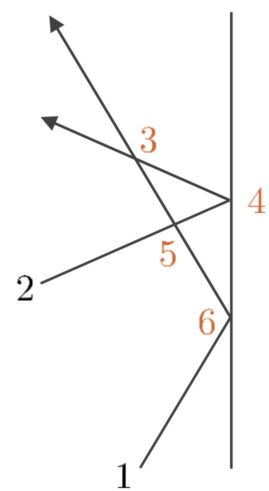
$L_{i,j}^{a,b} =$ 

$\in \mathrm{End}(F_{q^2})$

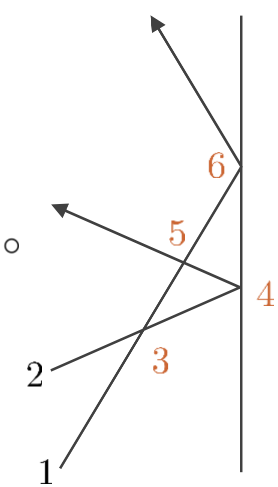
$$G = \begin{pmatrix} \mathbf{a}^+ & -p^{\frac{1}{2}}\mathbf{k} \\ p^{\frac{1}{2}}\mathbf{k} & \mathbf{a}^- \end{pmatrix} \in \mathrm{End}\left(V\otimes F_p\right)$$

$G_i^a =$ 

$\in \mathrm{End}(F_q)$



$\circ\,\mathcal{K}_{3456} = \mathcal{K}_{3456}\,\circ$



Reduction: General method

- QRE can be reduced and gives the solution of RE
  1. Consider the n-copies of QRE:

$$L_{1_j 2_j 3} G_{2_j 4} L_{2_j 1_j 5} G_{1_j 6} \mathcal{K}_{3456} = \mathcal{K}_{3456} G_{1_j 6} L_{1_j 2_j 5} G_{2_j 4} L_{2_j 1_j 3} \quad j = 1, \cdots, n$$

  2. Multiply $L_{1_j 2_j 3} G_{2_j 4} L_{2_j 1_j 5} G_{1_j 6}$ $(j = n, \cdots, 1)$ to $\mathcal{K}_{3456}$ from left and use QRE:

$$[L_{1_1 2_1 3} G_{2_1 4} L_{2_1 1_1 5} G_{1_1 6}] \cdots [L_{1_n 2_n 3} G_{2_n 4} L_{2_n 1_n 5} G_{1_n 6}] \mathcal{K}_{3456}$$
$$= \mathcal{K}_{3456} [G_{1_1 6} L_{1_1 2_1 5} G_{2_1 4} L_{2_1 1_1 3}] \cdots [G_{1_n 6} L_{1_n 2_n 5} G_{2_n 4} L_{2_n 1_n 3}]$$

$$\Leftrightarrow \quad [L_{1_1 2_1 3} \cdots L_{1_n 2_n 3}] [G_{2_1 4} \cdots G_{2_n 4}] [L_{2_1 1_1 5} \cdots L_{2_n 1_n 5}] [G_{1_1 6} \cdots G_{1_n 6}] \mathcal{K}_{3456}$$
$$= \mathcal{K}_{3456} [G_{1_1 6} \cdots G_{1_n 6}] [L_{1_1 2_1 5} \cdots L_{1_n 2_n 5}] [G_{2_1 4} \cdots G_{2_n 4}] [L_{2_1 1_1 3} \cdots L_{2_n 1_n 3}]$$

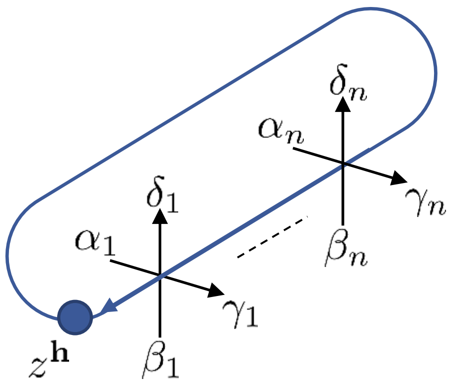
  3. Use $\mathcal{K}^{-1} = \mathcal{K}$ and $[(xy^{-1})^{\mathbf{h}_1} x^{\mathbf{h}_2} (xy)^{\mathbf{h}_3} y^{\mathbf{h}_4}, \mathcal{K}] = 0$, and take Tr over 3456:

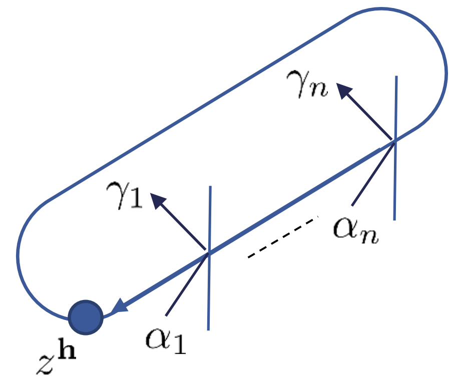
$$R_{\mathbf{12}}(xy^{-1}) K_{\mathbf{2}}(x) R_{\mathbf{21}}(xy) K_{\mathbf{1}}(y) = K_{\mathbf{1}}(y) R_{\mathbf{12}}(xy) K_{\mathbf{2}}(x) R_{\mathbf{21}}(xy^{-1})$$

$$R_{\mathbf{12}}(z) = \mathrm{Tr}_{F_{p^2}}(z^{\mathbf{h}_a} L_{1_1 2_1 a} \cdots L_{1_n 2_n a}) \in \mathrm{End}(V^{\otimes n} \otimes V^{\otimes n})$$
$$K_{\mathbf{1}}(z) = \mathrm{Tr}_{F_p}(z^{\mathbf{h}_a} G_{1_1 a} \cdots G_{1_n a}) \in \mathrm{End}(V^{\otimes n})$$

Reduction: 3dL and 3dG

$$\alpha_j, \beta_j, \gamma_j, \delta_j \in \{0,1\}$$





$$R(z)^{\gamma,\delta}_{\alpha,\beta} := \mathrm{Tr}\left(z^{\mathbf{h}} L^{\gamma_1,\delta_1}_{\alpha_1,\beta_1} \cdots L^{\gamma_n,\delta_n}_{\alpha_n,\beta_n}\right)$$

$$R(z) \in \mathrm{End}\left(V^{\otimes n} \otimes V^{\otimes n}\right)$$

$$K(z)^{\gamma}_{\alpha} = \mathrm{Tr}\left(z^{\mathbf{h}} G^{\gamma_1}_{\alpha_1} \cdots G^{\gamma_n}_{\alpha_n}\right)$$

$$K(z) \in \mathrm{End}\left(V^{\otimes n}\right)$$

- Natural Q: R and K have the origins of quantum groups?
- Yes! They are related to the fundamental reps of $U_q(A^{(1)}_{n-1})$. From now on, we fix $p = iq$.

$U_q(A_{n-1}^{(1)})$ and the fundamental reps

- Generators: $e_j, f_j, k_j^{\pm 1}\ (j \in \mathbb{Z}_n)$
- Cartan matrix: $a_{i,j} = 2\delta_{i,j} - \delta_{i,j+1} - \delta_{i,j-1}$
- Relations: $k_j k_j^{-1} = 1,\ [k_i, k_j] = 0,\ k_i e_j = q^{2a_{i,j}} e_j k_i,\ k_i f_j = q^{-2a_{i,j}} f_j k_i$

  $$[e_i, f_j] = \delta_{i,j} \frac{k_i - k_i^{-1}}{q^2 - q^{-2}} + \text{Serre type relations}$$

- Coproduct: $\Delta k_j^{\pm 1} = k_j^{\pm 1} \otimes k_j^{\pm 1},\ \Delta e_j = e_j \otimes 1 + k_j \otimes e_j,\ \Delta f_j = 1 \otimes f_j + f_j \otimes k_j^{-1}$
- Antipode: $S(k_j) = k_j^{-1},\ S(e_j) = -e_j k_k^{-1},\ S(f_j) = -k_j f_j$
- Fundamental rep $(W_{k,z}, \rho_{k,z})$

$$V^{\otimes n} = \oplus_{k=0}^{n} W_{k,z} \qquad V = \mathbb{C}v_0 \oplus \mathbb{C}v_1$$

$$W_{k,z} = \bigoplus_{\substack{(\alpha_1,\cdots,\alpha_n)\in\{0,1\}^n \\ \sum_{j=1}^{n} \alpha_j = k}} \mathbb{C}(q) v_{\alpha_1} \otimes \cdots \otimes v_{\alpha_n}$$

$$\rho_{k,z} : U_q \to \mathrm{End}(W_{k,z})$$

$$\begin{cases} \rho_{k,z}(k_j)v_\alpha = q^{2(\alpha_{j+1}-\alpha_j)} v_\alpha \\ \rho_{k,z}(e_j)v_\alpha = z^{\delta_{j,0}} v_{\alpha - \mathbf{e}_j + \mathbf{e}_{j+1}} \\ \rho_{k,z}(f_j)v_\alpha = z^{-\delta_{j,0}} v_{\alpha + \mathbf{e}_j - \mathbf{e}_{j+1}} \end{cases}$$

$$\mathbf{e}_j = (0, \cdots, \overset{j}{1}, \cdots, 0)$$

as $U_q(A_{n-1})$ module, $W_{k,z} \simeq V(\omega_k)$ i.e. k-th fundamental rep.

Characterization by quantum groups

- Prop [Bazhanov-Sergeev,06]
  - $R(z) := \mathrm{Tr}\left(z^{\mathbf{h}} L \cdots L\right) \in \mathrm{End}\left(V^{\otimes n} \otimes V^{\otimes n}\right)$ decomposes as

$$R(z) = \bigoplus_{0 \le l,m \le n} R_{l,m}(z), \quad R_{l,m}(xy^{-1}) \in \mathrm{End}(W_{l,x} \otimes W_{m,y})$$

  and satisfies the following intertwining relation

$$R_{lm}(xy^{-1}) \circ \rho_{l,x} \otimes \rho_{m,y}(\Delta(g)) = \rho_{l,x} \otimes \rho_{m,y}(\Delta^{\mathrm{op}}(g)) \circ R_{lm}(xy^{-1}) \quad {}^{\forall} g \in U_q(A^{(1)}_{n-1})$$

- What about $K(z) := \mathrm{Tr}\left(z^{\mathbf{h}} G \cdots G\right) \in \mathrm{End}\left(V^{\otimes n}\right)$?
- K-matrices are systematically constructed by considering some coideal subalgebra of U_q, parallel to R-matrices!

[Delius-Mackay,01]

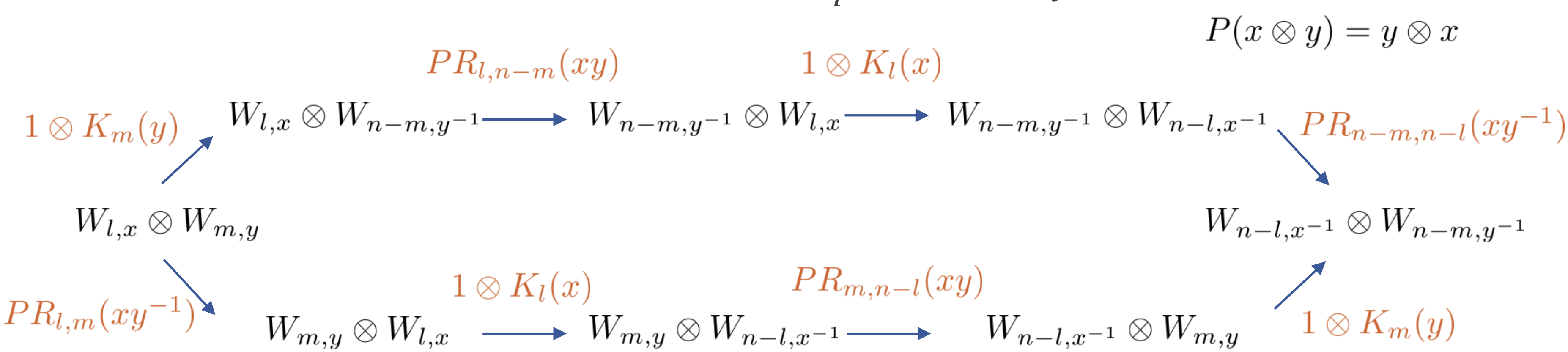
R-matrices	K-matrices
U_q	coideal subalg of U_q

Reflection equation and coideal subalg

- Let $\mathcal{J}_q$ be a left coideal subalg of $U_q(A_{n-1}^{(1)})$ i.e. $\Delta(\mathcal{J}_q) \subset U_q \otimes \mathcal{J}_q$
- Suppose $W_{l,z}$ and $W_{l,x} \otimes W_{m,y}$ be irreducible as $\mathcal{J}_q$-module
- Consider the intertwiner $K_l(z)\colon W_{l,z} \to W_{n-l,z^{-1}}$ s.t.

$$K_l(z) \circ \rho_{l,z}(g) = \rho_{n-l,z^{-1}}(g) \circ K_l(z) \quad {}^{\forall} g \in \mathcal{I}_p$$

- Let's consider the intertwiner of $\mathcal{J}_q$ in two ways



this gives the RE:

$$R_{\mathbf{12}}(xy^{-1})K_{\mathbf{2}}(x)R_{\mathbf{21}}(xy)K_{\mathbf{1}}(y) = K_{\mathbf{1}}(y)R_{\mathbf{12}}(xy)K_{\mathbf{2}}(x)R_{\mathbf{21}}(xy^{-1})$$

Generalized q-Onsager coideal

- Let's consider the specific coideal subalgebra $\mathcal{B}'_q$ generated by

$$b'_j = e_j + q^2 k_j f_j + \frac{iq}{1-q^2} k_j \quad (j \in \mathbb{Z}_n)$$

$$\Delta(b'_j) = k_j \otimes b'_j + (e_j - q^2 k_j f_j) \otimes 1 \quad \therefore \Delta(\mathcal{B}'_q) \subset U_q \otimes \mathcal{B}'_q$$

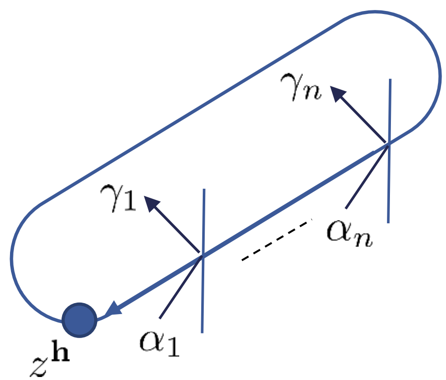
  - For $^\forall \alpha_j, \beta_j \in \mathbb{C}$, $e_j + \alpha_j k_j f_j + \beta_j k_j$ also generate left coideal subalgebras.
  - Our parameter choice
    - satisfies a necessary condition for nontrivial K.
    - corresponds to a type A generalized q-Onsager algebra [Baseilhac-Belliard,09] which is an example of quantum symmetric pair [Kolb,12].
  - $W_{l,z}$ and $W_{l,x} \otimes W_{m,y}$ are irreducible as $\mathcal{B}'_q$-module.

- Prop [Kuniba-Okado-Y, (in preparation)]
  - $K(z) := \mathrm{Tr}\left(z^{\mathbf{h}} G \cdots G\right) \in \mathrm{End}\left(V^{\otimes n}\right)$ decomposes as

$$K(z) = \bigoplus_{0 \le l \le n} K_l(z), \quad K_l(z) \in \mathrm{End}(W_{l,z} \to W_{n-l,z^{-1}})$$

and satisfy

$$K_l(z) \circ \rho_{l,z}(b) = \rho_{n-l,z^{-1}}(b) \circ K_l(z) \quad ^\forall b \in \mathcal{B}'_q$$



Outline

p doesn't appear

Is there some generalization?

- To summarize up to now… $L_{123}G_{24}L_{215}G_{16}\mathcal{K}_{3456} = \mathcal{K}_{3456}G_{16}L_{125}G_{24}L_{213}$
  1. QRE $\xrightarrow{\text{reduction}}$ MP solutions of RE
  2. Characterization in terms of quantum groups

- Any other R and K having MP structure?
  - Actually, for the symmetric tensor reps $(V_{l,z}, \pi_{l,z})$ $(l = 1,2,\cdots)$ of $U_q(A^{(1)}_{n-1})$, R-matrix has MP structure:

$$R(z) := \mathrm{Tr}_{F_{q^2}}\left(z^{-\mathbf{h}}\mathcal{R}\cdots\mathcal{R}\right) \in \mathrm{End}\left(F_{q^2}^{\otimes n} \otimes F_{q^2}^{\otimes n}\right)$$

$$\begin{cases} R(z) = \bigoplus_{l,m\in\mathbb{Z}_{\geq 0}} R_{l,m}(z), \quad R_{l,m}(xy^{-1}) \in \mathrm{End}(V_{l,x} \otimes V_{m,y}) & F_{q^2}^{\otimes n} = \oplus_{l\geq 0} V_{l,z} \\ R_{lm}(xy^{-1}) \circ \pi_{l,x} \otimes \pi_{m,y}(\Delta(g)) = \pi_{l,x} \otimes \pi_{m,y}(\Delta^{\mathrm{op}}(g)) \circ R_{lm}(xy^{-1}) & {}^{\forall}g \in U_q(A^{(1)}_{n-1}) \end{cases}$$

    Here, $\mathcal{R} \in \mathrm{End}(F_{q^2} \otimes F_{q^2} \otimes F_{q^2})$ is essentially the intertwiner of $A_{q^2}(SL_3)$ like $\mathcal{K}$ in $A_q(SP_4)$.

  - Natural Q: Is there ${}^{\exists}\mathcal{G} \in \mathrm{End}(F_{q^2} \otimes F_q)$ which gives K-matrices associated to the symmetric tensor reps by $K(z) = \mathrm{Tr}_{F_q}(z^{-\mathrm{h}}\mathcal{G}\cdots\mathcal{G}) \in \mathrm{End}(F_{q^2}^{\otimes n})$?

$U_q(A_{n-1}^{(1)})$ and the symmetric tensor reps

- Symmetric tensor rep $(V_{k,z}, \pi_{k,z})$ $\quad \pi_{k,z} : U_q \to \mathrm{End}(V_{k,z})$

$$F_{q^2}^{\otimes n} = \oplus_{k=0}^{\infty} V_k$$

$$V_{k,z} = \bigoplus_{(\alpha_1,\cdots,\alpha_n)\in B_k} \mathbb{C}(q^2)\,|\alpha_1,\cdots,\alpha_n\rangle$$

$$\begin{cases} \pi_{k,z}(k_j)\,|\alpha\rangle = q^{2(\alpha_{j+1}-\alpha_j)}\,|\alpha\rangle \\ \pi_{k,z}(e_j)\,|\alpha\rangle = z^{\delta_{j,0}}[\alpha_j]_{q^2}\,|\alpha - \mathbf{e}_j + \mathbf{e}_{j+1}\rangle \\ \pi_{k,z}(f_j)\,|\alpha\rangle = z^{-\delta_{j,0}}[\alpha_{j+1}]_{q^2}\,|\alpha + \mathbf{e}_j - \mathbf{e}_{j+1}\rangle \end{cases}$$

$$B_k = \{\alpha = (\alpha_1,\cdots,\alpha_n) \in \mathbb{Z}_{\geq 0}^n \mid \textstyle\sum_{j=1}^n \alpha_j = k\} \qquad [m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}$$

as $U_q(A_{n-1})$-module, $V_{k,z} \simeq V(k\omega_1)$

- $(V_{k,z}^*, \pi_{k,z}^*) :=$ the antipode dual rep of $V_{k,z}$ i.e. $V_{k,z}^* \simeq V(k\omega_{n-1})$

$$V_{k,z}^* = \bigoplus_{(\alpha_1,\cdots,\alpha_n)\in B_k} \mathbb{C}(q^2)\,|\alpha_1,\cdots,\alpha_n\rangle$$

$$\pi_{k,z}^* : U_q \to \mathrm{End}(V_{k,z}^*) \quad \begin{cases} \pi_{k,z}^*(k_j)\,|\alpha\rangle = q^{2(-\alpha_{j+1}+\alpha_j)}\,|\alpha\rangle \\ \pi_{k,z}^*(e_j)\,|\alpha\rangle = -\,z^{\delta_{j,0}}[\alpha_{j+1}+1]_{q^2}\,q^{2(\alpha_{j+1}-\alpha_j+2)}\,|\alpha + \mathbf{e}_j - \mathbf{e}_{j+1}\rangle \\ \pi_{k,z}^*(f_j)\,|\alpha\rangle = -\,z^{-\delta_{j,0}}[\alpha_j+1]_{q^2}\,q^{2(-\alpha_{j+1}+\alpha_j)}\,|\alpha - \mathbf{e}_j + \mathbf{e}_{j+1}\rangle \end{cases}$$

- We consider the problem in reverse order, i.e.

1. QRE $\xrightarrow{\text{reduction}}$ MP solutions of RE
2. Characterization in terms of quantum groups

Intertwining relation for K

- Let's consider the coideal subalgebra $\mathcal{B}_q$ generated by

$$b_j = e_j - q^4 k_j f_j - \frac{q^2}{1-q^2} k_j \quad (j \in \mathbb{Z}_n), \quad \Delta(\mathcal{B}_q) \subset U_q \otimes \mathcal{B}_q$$

  - $\mathcal{B}_q \simeq \mathcal{B}'_q \simeq$ generalized q-Onsager algebra
  - $V_{l,z}$ and $V_{l,x} \otimes V_{m,y}$ are irreducible as $\mathcal{B}_q$-module.

- Start from the intertwiner $K_l(z): V_{l,z} \to V^*_{l,z^{-1}}$

$$K_l(z) \circ \pi_{l,z}(b_j) = \pi^*_{l,z^{-1}}(b_j) \circ K_l(z) \quad (j \in \mathbb{Z}_n), \quad K(z)\,|\alpha\rangle = \sum_{\gamma \in B_l} K(z)^{\gamma}_{\alpha}\,|\gamma\rangle$$

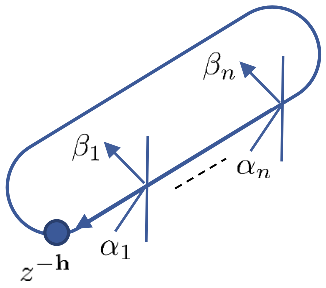
  This reads explicitly as

$$- z^{\delta_{j,0}}[\alpha_{j+1}]_{q^2} K(z)^{\gamma}_{\alpha+\mathbf{e}_j-\mathbf{e}_{j+1}} + z^{-\delta_{j,0}}[\alpha_j]_{q^2} q^{2(\alpha_j-\alpha_{j+1})} K(z)^{\gamma}_{\alpha-\mathbf{e}_j+\mathbf{e}_{j+1}} + \frac{1}{1-q^2} q^{2(\alpha_j-\alpha_{j+1}+1)} K(z)^{\gamma}_{\alpha}$$

$$= z^{-\delta_{j,0}}[\gamma_{j+1}]_{q^2} q^{2(-\gamma_j+\gamma_{j+1})} K(z)^{\gamma+\mathbf{e}_j-\mathbf{e}_{j+1}}_{\alpha} - z^{\delta_{j,0}}[\gamma_j]_{q^2} K(z)^{\gamma-\mathbf{e}_j+\mathbf{e}_{j+1}}_{\alpha} + \frac{1}{1-q^2} q^{2(-\gamma_j+\gamma_{j+1}+1)} K(z)^{\gamma}_{\alpha}$$

  This equation is only associated to j~j+1-th sites.

- Suppose $^\exists \mathcal{G} \in \mathrm{End}(F_{q^2} \otimes F_q)$ s.t. $K(z)^{\gamma}_{\alpha} = \mathrm{Tr}_{F_q}(z^{-\mathrm{h}} \mathcal{G}^{\gamma_1}_{\alpha_1} \cdots \mathcal{G}^{\gamma_n}_{\alpha_n})$.
  Here, $\mathcal{G}^j_i$ is given by $\mathcal{G}|i\rangle \otimes |m\rangle = \sum_{j \in \mathbb{Z}_{\geq 0}} |j\rangle \otimes \mathcal{G}^j_i |m\rangle$.



Bilinear equations for $\mathcal{G}$

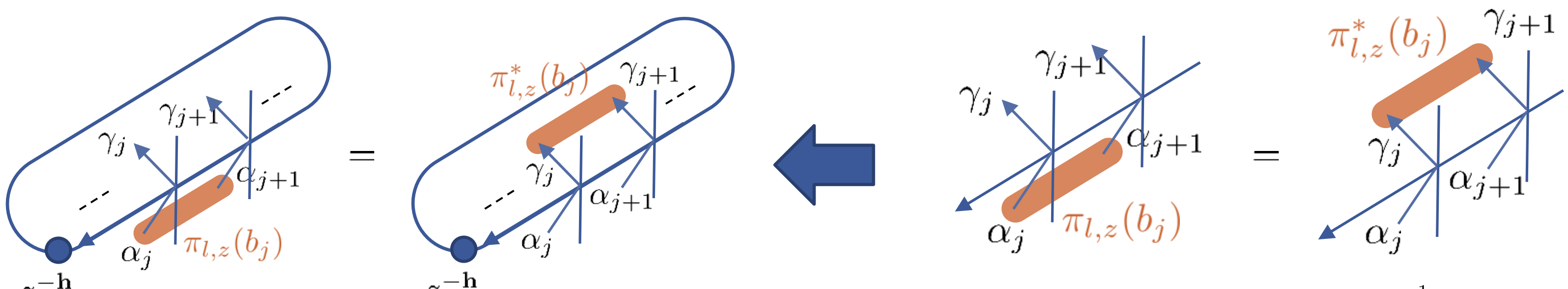
- Lemma [KOY,18]
  - If $\mathcal{G}_i^j$ satisfies the following equations

$$-[\alpha_2]_{q^2}\mathcal{G}_{\alpha_1+1}^{\gamma_1}\mathcal{G}_{\alpha_2-1}^{\gamma_2}+q^{2(\alpha_1-\alpha_2)}[\alpha_1]_{q^2}\mathcal{G}_{\alpha_1-1}^{\gamma_1}\mathcal{G}_{\alpha_2+1}^{\gamma_2}+\frac{1}{1-q^2}q^{2(\alpha_1-\alpha_2+1)}\mathcal{G}_{\alpha_1}^{\gamma_1}\mathcal{G}_{\alpha_2}^{\gamma_2}$$
$$=q^{2(-\gamma_1+\gamma_2)}[\gamma_2]_{q^2}\mathcal{G}_{\alpha_1}^{\gamma_1+1}\mathcal{G}_{\alpha_2}^{\gamma_2-1}-[\gamma_1]_{q^2}\mathcal{G}_{\alpha_1}^{\gamma_1-1}\mathcal{G}_{\alpha_2}^{\gamma_2+1}+\frac{1}{1-q^2}q^{2(-\gamma_1+\gamma_2+1)}\mathcal{G}_{\alpha_1}^{\gamma_1}\mathcal{G}_{\alpha_2}^{\gamma_2}$$

$\alpha_j,\gamma_j\in\mathbb{Z}_{\geq 0}$

then, $K(z)_\alpha^\gamma=\mathrm{Tr}(z^{-\mathrm{h}}\mathcal{G}_{\alpha_1}^{\gamma_1}\cdots\mathcal{G}_{\alpha_n}^{\gamma_n})$ satisfy the intertwining relation.

  - Sketch of proof



$$K_l(z)\circ\pi_{l,z}(b_j)=\pi^*_{l,z^{-1}}(b_j)\circ K_l(z)\quad(j\in\mathbb{Z}_n),$$

$$-[\alpha_2]_{q^2}\mathcal{G}_{\alpha_1+1}^{\gamma_1}\mathcal{G}_{\alpha_2-1}^{\gamma_2}+q^{2(\alpha_1-\alpha_2)}[\alpha_1]_{q^2}\mathcal{G}_{\alpha_1-1}^{\gamma_1}\mathcal{G}_{\alpha_2+1}^{\gamma_2}+\frac{1}{1-q^2}q^{2(\alpha_1-\alpha_2+1)}\mathcal{G}_{\alpha_1}^{\gamma_1}\mathcal{G}_{\alpha_2}^{\gamma_2}$$
$$=q^{2(-\gamma_1+\gamma_2)}[\gamma_2]_{q^2}\mathcal{G}_{\alpha_1}^{\gamma_1+1}\mathcal{G}_{\alpha_2}^{\gamma_2-1}-[\gamma_1]_{q^2}\mathcal{G}_{\alpha_1}^{\gamma_1-1}\mathcal{G}_{\alpha_2}^{\gamma_2+1}+\frac{1}{1-q^2}q^{2(-\gamma_1+\gamma_2+1)}\mathcal{G}_{\alpha_1}^{\gamma_1}\mathcal{G}_{\alpha_2}^{\gamma_2}$$

- The above equations = overdetermined recurrence equations

Explicit formula of $\mathcal{G}$

- Thm [KOY,18]
  - $\mathcal{G} \in \mathrm{End}(F_{q^2} \otimes F_q)$ is given by $\mathcal{G}_i^j = q^{-i^2}\mathbf{k}^{-2i} G_i^j$ and $(z;q) = \prod_{k=1}^{m}(1 - zq^{k-1})$

$$G_i^j = (-q^2;q^2)_{i+j}\,\phi\left(\begin{matrix} q^{-2j}, -q^{-2j} \\ -q^{-2i-2j} \end{matrix} ;q^2, q^2\mathbf{k}^2\right)(\mathbf{a}^-)^{i-j} \quad (i \geq j)$$

$$= (-q^2;q^2)_{i+j}(\mathbf{a}^+)^{i-j}\,\phi\left(\begin{matrix} q^{-2i}, -q^{-2i} \\ -q^{-2i-2j} \end{matrix} ;q^2, q^2\mathbf{k}^2\right) \quad (i \geq j)$$

Here, ϕ is q-hypergeometric series ${}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix} ;q, z\right) = \sum_{m\geq 0} \frac{(a;q)_m (b;q)_m}{(q;q)_m (c;q)_m} z^m$

  - $K(z)_\alpha^\gamma = \mathrm{Tr}_{F_q}(z^{-\mathrm{h}}\mathcal{G}_{\alpha_1}^{\gamma_1} \cdots \mathcal{G}_{\alpha_n}^{\gamma_n})$ decomposes as $K(z) = \bigoplus_{l\in\mathbb{Z}_{\geq 0}} K_l(z)$ and satisfy

$$K_l(z) \circ \pi_{l,z}(b) = \pi^*_{l,z^{-1}}(b) \circ K_l(z) \quad ({}^\forall b \in \mathcal{B}_q)$$

Sketch of proof for the formula of $\mathcal{G}$

- Substitute $\mathcal{G}_i^j$ into

$$-[\alpha_2]_{q^2}\mathcal{G}^{\gamma_1}_{\alpha_1+1}\mathcal{G}^{\gamma_2}_{\alpha_2-1} + q^{2(\alpha_1-\alpha_2)}[\alpha_1]_{q^2}\mathcal{G}^{\gamma_1}_{\alpha_1-1}\mathcal{G}^{\gamma_2}_{\alpha_2+1} + \frac{1}{1-q^2}q^{2(\alpha_1-\alpha_2+1)}\mathcal{G}^{\gamma_1}_{\alpha_1}\mathcal{G}^{\gamma_2}_{\alpha_2}$$

$$= q^{2(-\gamma_1+\gamma_2)}[\gamma_2]_{q^2}\mathcal{G}^{\gamma_1+1}_{\alpha_1}\mathcal{G}^{\gamma_2-1}_{\alpha_2} - [\gamma_1]_{q^2}\mathcal{G}^{\gamma_1-1}_{\alpha_1}\mathcal{G}^{\gamma_2+1}_{\alpha_2} + \frac{1}{1-q^2}q^{2(-\gamma_1+\gamma_2+1)}\mathcal{G}^{\gamma_1}_{\alpha_1}\mathcal{G}^{\gamma_2}_{\alpha_2}$$

  and remove some common factor, then it reduces to bilinear equations for q-hypergeometric series of scalar variables.

- For example, the case $\alpha_1 > \gamma_1$ and $\alpha_2 < \gamma_2$

$$\begin{aligned}
0 =& u_1(u_2-u_2^{-1})(-v_1^{-1};q)_2(q^{-1}u_1^2v_1^{-1}w;q)_2\phi\left(\begin{matrix}u_1,-u_1\\-q^{-1}v_1\end{matrix};w\right)\phi\left(\begin{matrix}qu_2,-qu_2\\-qv_2\end{matrix};y\right)\\
&+v_1^{-1}u_2(u_1v_1^{-1}-u_1^{-1}v_1)(-v_2^{-1};q)_2\phi\left(\begin{matrix}u_1,-u_1\\-qv_1\end{matrix};w\right)\phi\left(\begin{matrix}q^{-1}u_2,-q^{-1}u_2\\-q^{-1}v_2\end{matrix};y\right)\\
&-u_1v_2^{-1}(u_2v_2^{-1}-u_2^{-1}v_2)(-v_1^{-1};q)_2\phi\left(\begin{matrix}q^{-1}u_1,-q^{-1}u_1\\-q^{-1}v_1\end{matrix};w\right)\phi\left(\begin{matrix}u_2,-u_2\\-qv_2\end{matrix};y\right)\\
&-u_2(u_1-u_1^{-1})(-v_2^{-1};q)_2(q^{-1}u_1^2v_1^{-1}w;q)_2\phi\left(\begin{matrix}qu_1,-qu_1\\-qv_1\end{matrix};w\right)\phi\left(\begin{matrix}u_2,-u_2\\-q^{-1}v_2\end{matrix};y\right)\\
&-(1+q)u_1u_2(v_1^{-1}-v_2^{-1})(1+v_1^{-1})(1+v_2^{-1})(1-q^{-1}u_1^2v_1^{-1}w)\phi\left(\begin{matrix}u_1,-u_1\\-v_1\end{matrix};w\right)\phi\left(\begin{matrix}u_2,-u_2\\-v_2\end{matrix};y\right)
\end{aligned}$$

$u_1 = q^{-\gamma_1},\ u_2 = q^{-\alpha_2}$
$v_1 = q^{-\alpha_1-\gamma_1},\ v_2 = q^{-\alpha_2-\gamma_2}$
$w = q\mathbf{k},\ y = q^{\alpha_1+\alpha_2-\gamma_1-\gamma_2+1}\mathbf{k}$

Heine's contiguous relations →

$$0 = A\phi\left(\begin{matrix}q^{-1}u_1,-u_1\\-q^{-1}v_1\end{matrix};w\right) + B\phi\left(\begin{matrix}u_1,-q^{-1}u_1\\-q^{-1}v_1\end{matrix};w\right)\quad (A, B : \text{linear combination of } \phi)$$

Actually, we can show $A = B = 0$.

Example of K (n=2,l=2 case)

- We normalized $K_l(z)_\alpha^\gamma$ as $K_l(z)_{l\mathbf{e}_1}^{l\mathbf{e}_1} = 1$ $\quad K(z)\,|\alpha\rangle = \sum_{\gamma\in B_l} K(z)_\alpha^\gamma\,|\gamma\rangle$

$$|\alpha\rangle \mapsto |\gamma\rangle$$

$$|2,0\rangle \mapsto |2,0\rangle + \frac{(1+q^2)(1+q^4)z^2}{(q^2+z)(q^4+z)}\,|0,2\rangle + \frac{(1+q^2)z}{q^4+z}\,|1,1\rangle$$

$$|0,2\rangle \mapsto \frac{(1+q^2)(1+q^4)}{(q^2+z)(q^4+z)}\,|2,0\rangle + |0,2\rangle + \frac{1+q^2}{q^4+z}\,|1,1\rangle$$

$$|1,1\rangle \mapsto \frac{1+q^2}{q^4+z}\,|2,0\rangle + \frac{(1+q^2)z}{q^4+z}\,|0,2\rangle + \frac{z+q^6z+qz(2+z)+q^4(1+2z)}{(1+q^4)(q^2+z)(q^4+z)}\,|1,1\rangle$$

(not factorized)

- For ${}^\forall n \geq 2$, ${}^\forall l \in \mathbb{Z}_{\geq 0}$, $K_l(z)$ is trigonometric and dense.
- For $l = 1$ cases, our $K_1(z)$ reproduces [Gandenberger,99].

Some specialization for K

- $z = 1, q^{-2l}$
  - All the component of K-matrix are factorized for $\alpha, \gamma \in B_l$:

$$K(1)^{\gamma}_{\alpha} = \frac{\prod_{i=1}^{n}(-q^2;q^2)_{\alpha_i}(-q^2;q^2)_{\gamma_i}}{(-q^2;q^2)_l^2}, \quad K(q^{-2l})^{\gamma}_{\alpha} = \frac{q^{2\langle\gamma,\alpha\rangle}\prod_{i=1}^{n}(-q^2;q^2)_{\alpha_i+\gamma_i}}{(-q^2;q^2)_{2l}}$$

$\langle\gamma,\alpha\rangle = \sum_{1\le i<j\le n}\gamma_i\alpha_j$

- $q \to 0$
  - <u>Conjecture</u>

$$\lim_{q\to 0} K(z)^{\gamma}_{\alpha} = z^{-Q_0(\gamma,\alpha)}$$

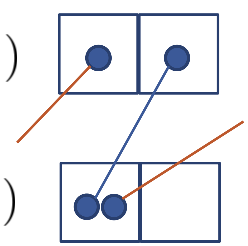
$Q_0(\gamma,\alpha) :=$ Nakayashiki-Yamada's unwinding number.

[Nakayashiki-Yamada,97]

  - <u>Example</u> (n=2,l=2)

$$|\alpha\rangle \mapsto |\gamma\rangle$$
$$|2,0\rangle \mapsto |2,0\rangle + |0,2\rangle + |1,1\rangle$$
$$|0,2\rangle \mapsto z^{-2}|2,0\rangle + |0,2\rangle + z^{-1}|1,1\rangle$$
$$|1,1\rangle \mapsto z^{-1}|2,0\rangle + |0,2\rangle + z^{-1}|1,1\rangle$$

$\alpha = (1,1)$



$\gamma = (2,0)$

$\to Q_0((2,0),(1,1)) = 1$

Quantized reflection equation for $\mathcal{G}$

- Does $\mathcal{G}$ satisfy QRE, like the following?

$$L_{123}G_{24}L_{215}G_{16}\mathcal{K}_{3456} = \mathcal{K}_{3456}G_{16}L_{125}G_{24}L_{213} \in \mathrm{End}(\overset{1}{V} \otimes \overset{2}{V} \otimes \overset{3}{F_{p^2}} \otimes \overset{4}{F_p} \otimes \overset{5}{F_{p^2}} \otimes \overset{6}{F_p})$$

- Theorem [KOY, (in preparation)]
  - The following equations holds:

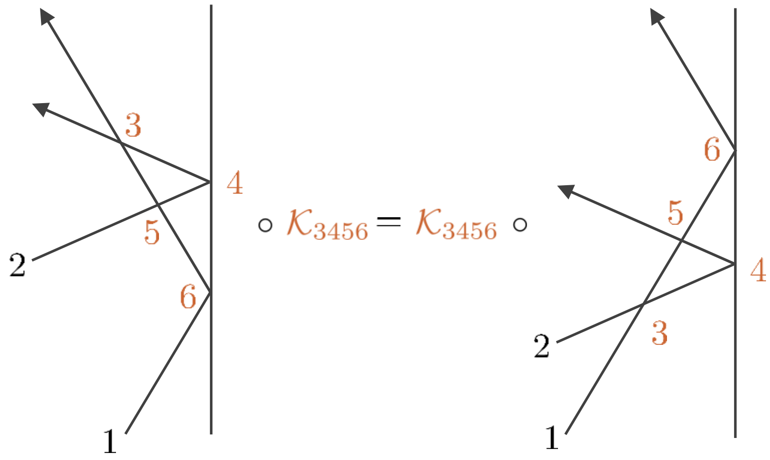
$$\mathcal{R}_{123}\widetilde{\mathcal{G}}_{24}\widetilde{\mathcal{R}}_{215}\widetilde{\mathcal{G}}_{16}\mathcal{K}_{3456} = \mathcal{K}_{3456}\widetilde{\mathcal{G}}_{16}\widetilde{\mathcal{R}}_{125}\widetilde{\mathcal{G}}_{24}\overline{\mathcal{R}}_{213} \in \mathrm{End}(\overset{1}{F_{q^2}} \otimes \overset{2}{F_{q^2}} \otimes \overset{3}{F_{q^2}} \otimes \overset{4}{F_q} \otimes \overset{5}{F_{q^2}} \otimes \overset{6}{F_q})$$

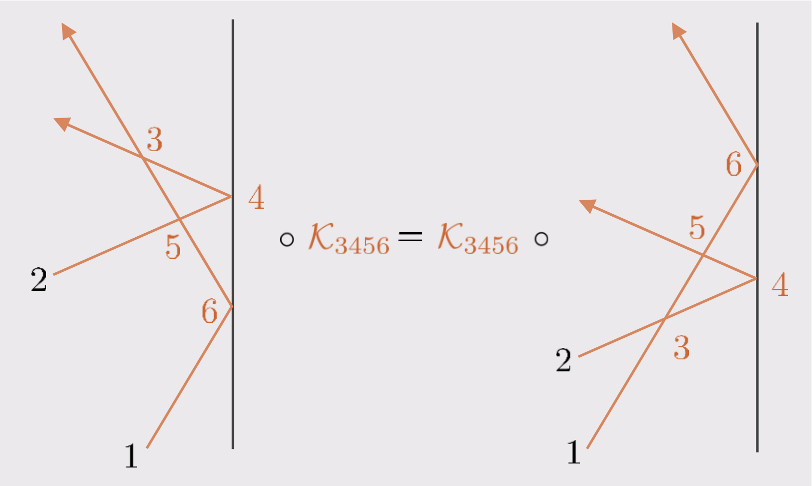
$$\tilde{\mathcal{G}}_i^j = \mathrm{q}^{-\mathbf{h}^2/2}\mathcal{G}_i^j\mathrm{q}^{\mathbf{h}^2/2}$$

$\mathcal{R}, \tilde{\mathcal{R}}, \bar{\mathcal{R}} \simeq$ the intertwiner of $A_{q^2}(SL_3)$

—— : $V = \mathbb{C}v_0 \oplus \mathbb{C}v_1$

—— : $F = \oplus_{m\geq 0}\mathbb{C}\,|m\rangle$





Summary and Outlook

- The fundamental reps

$$L_{123}G_{24}L_{215}G_{16}\mathcal{K}_{3456}$$
$$= \mathcal{K}_{3456}G_{16}L_{125}G_{24}L_{213}$$
$$\downarrow$$
$$K(z) = \mathrm{Tr}\left(z^{\mathbf{h}}G\cdots G\right)$$
$$\downarrow$$
$$K_l(z)\circ\rho_{l,z}(b) = \rho_{n-l,z^{-1}}(b)\circ K_l(z)$$

- The symmetric tensor reps

$$\mathcal{R}_{123}\widetilde{\mathcal{G}}_{24}\widetilde{\mathcal{R}}_{215}\widetilde{\mathcal{G}}_{16}\mathcal{K}_{3456}$$
$$= \mathcal{K}_{3456}\widetilde{\mathcal{G}}_{16}\widetilde{\mathcal{R}}_{125}\widetilde{\mathcal{G}}_{24}\overline{\mathcal{R}}_{213}$$
$$\uparrow$$
$$K(z) = \mathrm{Tr}\left(z^{-\mathbf{h}}\mathcal{G}\cdots\mathcal{G}\right)$$
$$\uparrow$$
$$K_l(z)\circ\pi_{l,z}(b) = \pi_{n-l,z^{-1}}(b)\circ K_l(z)$$

- Boundary reduction known for $(L, G, \mathcal{K})$: [Kuniba-Pasquier,18]

$$R^{r,r'}(z) := \langle\chi_r|z^{\mathbf{h}}L\cdots L|\chi_{r'}\rangle \qquad K^{s,s'}(z) := \langle\eta_s|z^{\mathbf{h}}G\cdots G|\eta_{s'}\rangle$$

$$|\chi_s\rangle = \sum_{m\geq 0}\frac{|sm\rangle}{(q^{2s^2})_m}$$

They conjecturally satisfy the RE in the following pairs

$$|\eta_s\rangle = \sum_{m\geq 0}\frac{|sm\rangle}{(q^{s^2})_m}$$

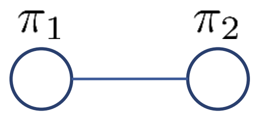
$r, r', s, s' \in \{1,2\}$

R	K	$U_q(\hat{\mathfrak{g}})$
$R^{1,1}(z)$	$K^{1,1}(z)$, $K^{1,2}(z)$, $K^{2,1}(z)$, $K^{2,2}(z)$	$U_q(D^{(2)}_{n+1})$
$R^{2,1}(z)$	$K^{2,1}(z)$, $K^{2,2}(z)$	$U_q(B^{(1)}_n)$
$R^{2,2}(z)$	$K^{2,2}(z)$	$U_q(D^{(1)}_n)$

How about $(\mathcal{R}, \tilde{\mathcal{G}}, \mathcal{K})$?

Appendix: $A_q(SL_3)$

$A_q(SL_3) = \langle t_{ij} \rangle_{i,j=1}^{3}$



$$\begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} \mapsto \overset{\pi_1}{\begin{pmatrix} \mathbf{a}^- & \mathbf{k} & 0 \\ -q\mathbf{k} & \mathbf{a}^+ & 0 \\ 0 & 0 & 1 \end{pmatrix}}, \quad \overset{\pi_2}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathbf{a}^- & \mathbf{k} \\ 0 & -q\mathbf{k} & \mathbf{a}^+ \end{pmatrix}}$$

$W(sl_3) = \langle s_1, s_2 \rangle \qquad s_1 s_2 s_1 = s_2 s_1 s_2$

$\Rightarrow \quad \pi_1 \otimes \pi_2 \otimes \pi_1 \simeq \pi_2 \otimes \pi_1 \otimes \pi_2$

- By Schur's Lemma, there exists the unique intertwiner Φ s.t.

$$\Phi \circ \pi_1 \otimes \pi_2 \otimes \pi_1(\Delta(t_{ij})) = \pi_2 \otimes \pi_1 \otimes \pi_2(\Delta(t_{ij})) \circ \Phi$$

Appendix: 3dR

- Set $\mathcal{R} = \Phi P_{13} \in \mathrm{End}\left((F_q)^{\otimes 3}\right)$ $\quad P_{13}(x \otimes y \otimes z) = z \otimes y \otimes x$

  $$\mathcal{R} \circ \pi_1 \otimes \pi_2 \otimes \pi_1(\Delta^{\mathrm{op}}(t_{ij})) = \pi_2 \otimes \pi_1 \otimes \pi_2(\Delta(t_{ij})) \circ \mathcal{R} \in \mathrm{End}\left((F_q)^{\otimes 3}\right)$$

  Here, $\Delta^{\mathrm{op}}(t_{ij}) = P_{13}\Delta(t_{ij})P_{13}$

- Denote its component as $\mathcal{R}\,|i\rangle \otimes |j\rangle \otimes |k\rangle = \sum_{a,b,c} \mathcal{R}^{abc}_{ijk}\,|a\rangle \otimes |b\rangle \otimes |c\rangle$

- Prop [KV94]
  - With the normalization $\mathcal{R}\,|0\rangle \otimes |0\rangle \otimes |0\rangle = |0\rangle \otimes |0\rangle \otimes |0\rangle$

  $$\mathcal{R}^{a,b,c}_{i,j,k} = \delta^{a+b}_{i+j}\delta^{b+c}_{j+k} \sum_{\lambda,\mu \geq 0, \lambda+\mu=b} (-1)^{\lambda} q^{i(c-j)+(k+1)\lambda+\mu(\mu-k)} \frac{(q^2)_{c+\mu}}{(q^2)_c} \binom{i}{\mu}_{q^2} \binom{j}{\lambda}_{q^2}$$

  $$\binom{a}{b}_q = \frac{(q)_a}{(q)_b (q)_{a-b}}, \quad (q)_k = \prod_{l=1}^{k} (1-q^l)$$

  - Let $\mathcal{R}\,|i\rangle \otimes |j\rangle \otimes |k\rangle = \sum_{a,b} |a\rangle \otimes |b\rangle \otimes \mathcal{R}^{a,b}_{i,j}\,|k\rangle$

  $$\mathcal{R}^{a,b}_{i,j} = \delta^{a+b}_{i+j} \sum_{\lambda+\mu=b} (-1)^{\lambda} q^{\lambda+\mu^2-ib} \binom{i}{\mu}_{q^2} \binom{j}{\lambda}_{q^2} (\mathbf{a}^-)^{\mu} (\mathbf{a}^+)^{j-\lambda} \mathbf{k}^{i-\lambda-\mu}$$

Appendix: Tetrahedron equation

- Theorem [KV94]
  - $\mathcal{R}$ satisfies the tetrahedron eq.

$$\mathcal{R}_{124}\mathcal{R}_{135}\mathcal{R}_{236}\mathcal{R}_{456} = \mathcal{R}_{456}\mathcal{R}_{236}\mathcal{R}_{135}\mathcal{R}_{124} \in \mathrm{End}\left(F_q^{\otimes 6}\right)$$

- Sketch of Proof
  - Consider $A_q(sl_4)$ and $W(sl_4) = \langle s_1, s_2, s_3 \rangle$
  - TE is given by considering the intertwiner for $s_1 s_2 s_3 s_1 s_2 s_1 = s_3 s_2 s_3 s_1 s_2 s_3$ in 2 different ways. Here we use the following intertwiners:

isomorphism	intertwiner
$\pi_1 \otimes \pi_3 \simeq \pi_3 \otimes \pi_1$	P_{12}
$\pi_1 \otimes \pi_2 \otimes \pi_1 \simeq \pi_2 \otimes \pi_1 \otimes \pi_2$	$\Phi = \mathcal{R}P_{13}$
$\pi_2 \otimes \pi_3 \otimes \pi_2 \simeq \pi_3 \otimes \pi_2 \otimes \pi_3$	$\Phi = \mathcal{R}P_{13}$

- TE is 3d analog of YBE and gives the sufficient condition for the commutativity of layer-to-layer transfer matrix.